THE TEXT IS LIGHT IN THE BOOK

L. ROSENFELD

NUCLEAR FORCES

MONOGRAPHS ON THEORETICAL AND APPLIED PHYSICS

I

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NUCLEAR FORCES

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TO THE MEMORY OF
JACQUES SOLOMON

Πρὸ γόων δὲ μνᾶστις
SIMONIDES

PREFACE

...χρεὼ δέ σε πάντα πυθέσθαι
ἠμὲν Ἀληθείης εὐκυκλέος ἀτρεμὲς ἦτορ
ἠδὲ βροτῶν δόξας, ταῖς οὐκ ἔνι πίστις ἀληθής.

PARMENIDES. *Περὶ φύσεως*, Fragm. *1*, 28-30.

Now shalt thou hear of all things, of well-rounded Truth's unmoved heart and of mortals' opinions, in which there is no true assurance.

The present book originated in an attempt to survey the available empirical data on various properties of nuclear systems with a view to deriving from it as much unambiguous information as possible on the character of the specific forces acting between nucleons. This enquiry had not proceeded very far, however, before it became clear that, in order to make it really useful, its scope had to be greatly enlarged. At first, I had imagined that it would have been possible to limit myself to a theoretical analysis of the experimental data without entering into detail about the methods by which such data were obtained. But I was soon confronted with the difficulty that no comprehensive and up-to-date account of the experiments was or could be expected in the next future to become available because of the rapid and unsystematic accumulation of material which had taken place during the last decade. Being forced to wade through this luxuriant but often swampy growth, I thought it would be useful to give the reader the benefit of this harassing experience, by incorporating into the account the critical surveys of different aspects of nuclear research which I had first attempted to make for my own information. It was clear that such a plan would not only involve a considerable increase in the size of the book, but would also tend to give it a somewhat hybrid character: on the one hand, it would retain its original purpose of developing a definite line of theoretical argument; on the other, by its presentation of a fairly complete summary of the experimental data, it would acquire the ephemeral quality of a reference book. This raised a problem of composition, of whose difficulties I was only too well aware; but it seemed to me that it had to be faced, as it appeared to correspond to the actual needs of the present situation.

In accordance, therefore, with the original plan, the arrangement and analysis of the material follows an inductive line of approach; but detailed, and accordingly lengthy, accounts of both experimental data and theoretical methods have been inserted at the appropriate places. This not only makes the progress of the discussion much slower, but even threatens to conceal the fairly simple thread of the argument running through the whole book; the latter defect I have tried to remedy by inserting wherever necessary

introductory remarks and summaries. Another drawback of the disposition adopted is that certain aspects of the subject have come to be treated in successive instalments, separated from each other by other considerations: for instance, the theory of the meson field is not presented in all its complication at once, but the various refinements are gradually introduced in immediate connexion with the features of nuclear forces which render them necessary or desirable. This is admittedly unsatisfactory for those readers who are looking for a self-contained survey of just one particular aspect. I have endeavoured to provide to some extent for this need by multiplying cross-references and extending the subject index. But keeping in mind the main purpose of the book, it was inevitable that the wish to bring out more sharply the relations of the different stages of the theoretical analysis to the experimental evidence should have overcome the exigencies of "elegance".

Although the general set-up is theoretical, and the emphasis laid throughout on the theoretical problems, I hope, also, that the experimental physicist — confining himself, if he so wishes, to those sections where the density of formulae is least — may find the discussion of some use. The theoretical treatment itself does not require more than elementary quantum mechanics; even so, it is in most cases fairly complete, except of course with regard to meson field theory, and in a few instances where reference is made to original papers for a fuller account of secondary points. I have always had in mind the advanced student, wishing to obtain a general view of the subject with concise introductions to its several parts, before engaging in research on some specific question. I have accordingly taken pains to unify and harmonize as much as possible the scattered contributions from various authors, the mushroom-like proliferation of which offers a bewildering variety in the choice of basic assumptions, numerical values of parameters, notations, and even terminology. I have, however, renounced unification in some cases where it would have involved an amount of numerical work out of proportion to the significance of the result. Likewise, it was not practicable to adhere strictly to the golden rule of using any letter or symbol in only one sense throughout; nevertheless it is hoped that no confusion is likely to arise in this respect. As regards terminology, I have been, of course, very sparing in the introduction of new terms; I have only to apologise for two neologisms: the one is the phrase *dichotomic variable*, adopted for didactic purposes to denote (roughly speaking) quantities with only two eigenvalues; the other is the collective name *lepton*, applied to particles of small mass (electrons and neutrinos). The need for such a collective term seems to be widely felt, and I hope that the word here proposed will meet with general acceptance; it was coined by Prof. Møller in consultation with a hellenist — a most happy instance of fruitful collaboration between science and humanities. Finally, it must be mentioned that in the compilation of the tables of empirical data, or in quoting results of numerical calculations, I was

occasionally confronted with the delicate problem of correcting the published values; it has not been found worth while, however, to make a special note of all cases in which the value appearing in this book differs slightly from that given in the original publication: the *very* distrustful reader can always check the figures if he likes.

It will not be necessary here to explain in more detail the lines along which the programme outlined above has been carried out; the main points are set forth in the Introduction. The most obviously objectionable feature of the book is undoubtedly its size, which, considering its limited scope, is certainly a bad sign. Referring to the motto borrowed from Parmenides' poem, the reader will indeed find that much more space is taken by uncertain recounting of "mortals' opinions" than by proud disclosure of "well-rounded truth"; and that our picture of nuclear forces is still very far from the ideal attained in atomic theory, where so very simple and neat a system of concepts and laws suffices to account, according to Dirac's famous phrase *, for "a large part of physics and the whole of chemistry". However, let this prove an incentive to youthful readers to try and give shape to new ideas suited to bring us salvation. In any event, I feel no regret that this laborious quest, conducted through so many heavy pages, should not end on any clear-cut conclusion. It is actually my warmest wish (a wish *not* shared by my publisher) that Part IV especially, which deals with non-central forces, should rapidly become still more obsolete than it already is.

Throughout the various stages of preparation of this work, I have been fortunate in enjoying most generous help and constructive criticism from many quarters. From the period of the first drafting, which carries us back to the chilly and famine-stricken Utrecht of the winter 1944–45, I remember, not without emotion, many discussions with my friends Opechowski and Lubanski, in which we found such comfort in those distressing times. Lubanski's premature death will long be felt as a great loss by those who had the opportunity of witnessing the unfolding of his talent and of appreciating his fine personality. Unpublished material of fundamental importance for the argument of the book has obligingly been provided by Dr. Hulthén, Mr. de Jager, Mr. Ramsey and, above all, Dr. Fröhlich and Dr. Powell; without this help it would not have been possible to treat in any reasonably adequate way the problems occupying large portions of Chapters VII, VIII and XIV. My thanks are due to Prof. Leprince–Ringuet, Dr. Powell and Dr. Roberts for supplying photographs for reproduction. The drafts of the figures, with the exception of a few prepared by Mr. D. J. Bouman, have been supplied by Mr. A. Wapstra. The figures themselves have been very skilfully drawn by the publisher's draughtsman, Mr. den Uyl, who took up his task with great devotion and industry; he contributed many helpful suggestions in difficult cases. To

* P. DIRAC, *Proc. Roy. Soc.* **A 123**, 714, 1929.

Mr. Wapstra I am also indebted for the compilation of the large table of nuclei, embodying the latest available data *. Part of the manuscript was weeded of barbarisms by Miss S. Power and the Rev. O. Walsh, and this task was later completed on the proof by Mr. Huby; neither of these obliging helpers, however, can be held responsible for any linguistic errors which may have been introduced in last minute alterations. A proof has also been read with great care by Dr. Fröhlich, Prof. Heitler and Prof. Møller; to all of them I owe many important remarks and corrections. I have reserved to the last, however, mention of the collaborator to whom I am in the greatest debt: during the long drawn out process of preparing the manuscript for the press and proof-reading, Dr. Podolanski has devoted his whole time and energy to this tiresome task. It is impossible to assess the improvements which every page of the book has derived not only from his first-hand experience of typographical work, but also from his painstaking scrutiny of the text. His untiring patience and cheerfulness in the face of difficulties have been of no little comfort to me under sometimes trying circumstances. It is fitting to close this long list of acknowledgments with a word of praise for the publishers, who have bravely carried their undertaking to a good end in a most unpropitious time.

* Since this table was prepared only after the completion of the text, no references to it could be inserted to replace those to the tables of Mattauch and Flügge.

Manchester, December 1947.

NOTATIONS

N.1. Mathematical symbols

N.10. The following list includes such symbols as are not explained in the text, while their use is not universally standardized.

N.11. *Numbers and observables.* If A, B are any two numbers or (whenever this more general meaning is possible) two observables (operators),

$A \ldots B$ denotes the interval between A and B, or some number in this interval;
A^* denotes the complex conjugate of A;
$\Re A$ the real part of the complex quantity A;
$A^\dagger$ the adjoint of the operator A;
$\log A$ the natural logarithm of A;
$\operatorname{tr} A$ the trace of the operator A;
$\operatorname{av} A$ the expectation value (average value, mean value) of the observable A in some specified state;
$[A, B]_-$, or simply $[A, B]$, the commutator $AB - BA$;
$[A, B]_+$ the anticommutator $AB + BA$.

N.12. *Vectors and tensors.* When the tensor notation involving covariant and contravariant components is used, the usual rule of summation over indices occurring twice is adopted; the dummy index then occupies a covariant and a contravariant position.

Ordinary space vectors are denoted by arrows; the *vector product* of the vectors $\vec{a}, \vec{b}$ is represented by $\vec{a} \wedge \vec{b}$.

The symbol (cycl) following a relation which involves vector components means that the relations derived from that one by cyclic permutation of the vector components also hold.

N.13. *Functions of dynamical variables.* The letter Q denotes in a general way some set of coordinates pertaining to a particle; if we deal with a system of identical particles, distinguished by numbers 1, 2, ..., i, ..., all quantities pertaining to the i-th particle are distinguished by the upper index (i), e.g. $Q^{(i)}$.

If A, B are any functions of dynamical variables,
$A \approx B$ means A is approximately equal to B, or A is of the same order of magnitude as B,

$A \sim B$ means A is proportional to B,
$A \simeq B$ means A has asymptotic form B;
$\dot{A}$ denotes the derivative of A with respect to ct, t being the time variable,
$\triangle A$ the Laplacian operator $\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$,
$\Box A$ the Dalembertian operator $\triangle A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$.

Volume integration is commonly denoted by $\int \ldots dv$, the volume element $dx\,dy\,dz$ being represented by dv. If one wishes to specify the volume element attached to point P, one writes dv_P; for that pertaining to the extremity of the radius vector $\vec{\xi}$, one writes dv_ξ. According to this system of notation, the element of momentum space (the momentum being $\vec{p}$) will be represented by dv_p. The element of solid angle (in any space) is denoted by $d\Omega$.

A symbol of integration not accompanied by any symbol for a volume element represents an operation involving both summations over certain coordinates and integrations over other.

The Dirac δ-function with a vector as argument is used in the sense

$$\delta(\vec{k}-\vec{k}') = (2\pi)^{-3}\int e^{i(\vec{k}-\vec{k}')\vec{x}}\,dv.$$

N.14. *Legendre polynomials and tesseral harmonics.* A very convenient summary of the most useful facts and formulae of the theory of Legendre polynomials and tesseral harmonics is given by BETHE [33]. Following his example, we adopt Darwin's proposal of a uniform definition of Y_l^m for m-values of both signs. We list here for reference the general definitions:

$$P_l(x) = \frac{1}{2^l \cdot l!}\cdot\frac{d^l}{dx^l}(x^2-1)^l$$

$$Y_l^m(\vartheta,\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}\cdot\sqrt{\frac{2l+1}{2}\cdot\frac{(l-m)!}{(l+m)!}}\sin^m\vartheta\left[\frac{d^m}{dx^m}P_l(x)\right]_{x\equiv\cos\vartheta}$$

and the simplest special cases:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}};$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\vartheta, \qquad Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\vartheta\,e^{i\varphi};$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\vartheta - \tfrac{1}{2}\right),\ Y_2^1 = \sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta\,e^{i\varphi},\ Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\vartheta\,e^{2i\varphi}.$$

N.2. System of references

N.21. *Numbering of sections, etc.* Each Chapter of the book is divided into sections and the latter are generally further subdivided. The subsections receive a decimal numbering, in which the first digit on the right of the dot refers to the section, and the following digits to the successive subdivisions of this section; on the left of the dot appears either a number which is that of the Chapter (with **0** for the Introduction), or (for the preliminary sections) a letter **N** (Notations) or **U** (Units and Constants), or a composite sign such as **A2**, referring to the Appendices. Tables and figures occurring in any subsection are designated by the number of that subsection, followed if necessary by an ordinal number; e.g.: fig. **3.221–2**, table **6.22–1**. Formulae are numbered through *in each section separately*; within the section they are referred to simply by their ordinal number between brackets. References to formulae of other sections include also the number of the *subsection* in which they occur; e.g.: formula (*4.331–22*). (Strictly, the number of the section would be sufficient, but the more precise indication of the subsection facilitates their location.)

N.22. *References to literature.* Reference to a book or paper is given by the name of the author, followed by the last two digits of the year of publication between square brackets; the necessary details will then be found in the author index at the end of the book, which is alphabetical with respect to authors' names and chronological for each author *. Several papers published within the same year are distinguished by letters *a*, *b*, *c*, ... In case of joint authorship, the paper is classified under the first name quoted; it has sometimes be found advisable to deviate from the order of names adopted by the authors themselves. E.g. the reference BREIT [37*b*] concerns a paper by S. SHARE and G. BREIT, which is more conveniently listed together with other papers by Breit and his collaborators, relating to similar subjects.

The abbreviations used to designate the scientific journals are the traditional ones, except in a few cases, which are explained at the beginning of the author index. Besides, some repeatedly quoted books or articles will be denoted in the text by the following signs:

B & B, H. BETHE and R. BACHER, *Rev. Mod. Phys.* **8**, 82. 1936.

M & F, J. MATTAUCH and S. FLÜGGE, *Kernphysikalische Tabellen* (1941). This work is supplemented by a later article:

M & F*, S. FLÜGGE and J. MATTAUCH, *Physik. Z.* **44**, 181, 1943 (with a correction *ibid.*, p. 391).

W & W, E. WHITTAKER and G. WATSON, *A course of modern analysis* (4th Ed., 1935).

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* Some incidental references are directly given in full in footnotes.

UNITS AND CONSTANTS

U.1. Units

U.11. *Relativistic correlation of units.* According to the conceptions of relativity theory, a correlation can be established between the units of length and time on the one hand, momentum and energy, and mass and energy on the other, simply by taking the velocity of light as unit of velocity. The current practice of nuclear physics coincides only in part with this rational choice. While it is customary to measure masses and momenta in energy units, one keeps, e.g., to the second as the unit of time. In order to conform as far as possible to such usage, the following compromise has been resorted to:

(*a*) Mass and momentum are given the dimension of an energy and measured in energy units; i.e. the mass of a particle is identified with its rest-energy, while we call momentum the usual quantity of that name multiplied by the velocity of light.

(*b*) This convention involves similar modifications of derived quantities: an angular momentum will likewise be multiplied by c, a moment of inertia by c^2; in particular, we take as fundamental constant of "action" Dirac's constant $\hbar$ multiplied by c, which we denote by $\hbar$. In order to uphold an expression of the form $\hbar\nu$ for a quantum of energy, we characterize periodic changes (whose velocity of propagation is c) by their circular wave-number ν, i.e. the quotient of their circular frequency by c.

(*c*) On the other hand, we maintain the second as unit of time and (unless explicitly stated otherwise) the cm/sec as unit of velocity, so that the velocity of light will still appear explicitly in formulae involving times or velocities.

U.12. *Mass and energy units.* In most cases, the mega-electronvolt, MeV, is a convenient unit in which to express masses as well as energies and momenta. However, it may sometimes be advantageous to compare energies to the electron mass m, or masses to the physical mass unit, MU, defined as $^1/_{16}$ of the mass of the oxygen *atom* ^{16}O. The conversion factors are as follows (mMU denoting the milli-mass-unit):

$$\begin{aligned} 1\ \text{mMU} &= 0{,}93\ \ \text{MeV} \\ 1\ \text{MeV} &= 1{,}075\ \text{mMU} \\ m &= 0{,}51\ \ \text{MeV} = 0{,}548\ \text{mMU}. \end{aligned}$$

U.2. Constants

U.21. *Fundamental constants.* We collect in the following table the values of the fundamental constants establishing the correlation between atomic and nuclear dimensions, and between these dimensions and the macroscopic standards, together with some useful derived constants. The accuracy is limited to the present requirements of nuclear physics.

U.21. *Fundamental constants*

Constant	Symbol	Value
Relations to c.g.s. standards		
Velocity of light	c	$3 \cdot 10^{10}$ cm sec^{-1}
Quantum of action	$\hbar$	$1{,}97 \cdot 10^{-5}$ eV cm
Compton wave-length *	$\hbar/m$	$0{,}386 \cdot 10^{-10}$ cm
Boltzmann's constant	k	0,86 MeV/10^{10} degrees
Dimensionless constants		
Fine structure constant	$e^2/\hbar$	1/137
Mass ratio proton-electron	M_p/m	1836
Derived constants		
Electron radius	$d = e^2/m$	$2{,}82 \cdot 10^{-13}$ cm
Relativistic proton wave-length *	$\hbar/M_p$	$0{,}21 \cdot 10^{-13}$ cm
Inverse thereof *	$M_p/\hbar$	$4{,}76 \cdot 10^{13}$ cm^{-1}
Inverse Compton wave-length *	$m/\hbar$	$2{,}59 \cdot 10^{10}$ cm^{-1}
Nuclear Bohr magneton	$\mu_0 = e\hbar/2M_p$	$3{,}1 \cdot 10^{-11}$ eV/gauss

* Wave-length divided by 2π.

U.22. *Nuclear constants.* It will be convenient to collect here the values of the parameters characterizing the main properties of nucleons and of the deuteron, which are among the best established nuclear constants.

U.22. *Nuclear constants*

Constant	Symbol	Value	Reference
Nucleons			
Mass difference neutron-proton	$M_n - M_p$	$2{,}47\,m$	(*1.22*–5)
Magnetic moment of proton	μ_p	2,7896	(*1.21*–3)
neutron	μ_n	−1,9103	(*1.22*–6)
proton + neutron	$\mu_d^0 = \mu_p + \mu_n$	0,8793	(*6.12*–10)
Deuteron			
Magnetic moment	μ_d	0,8565	(*6.12*–9)
Admixture of D state to ground state	$\sin^2\omega$	0,04	(*6.12*–12)
Electric quadrupole moment	Q	$2{,}73 \cdot 10^{-27}$ cm^2	(*6.12*–7)
Binding energy of ground state	$\lvert\varepsilon_0\rvert$	2,185 MeV	(*6.11*–1)
Energy of ^1S virtual level	$^1\varepsilon(0)$	0,065 MeV	*6.431*
Proton-neutron scattering radii (for zero range)	$^3a_0$	$0{,}437 \cdot 10^{-12}$ cm	*6.431*
	$^1a_0$	$-2{,}47 \cdot 10^{-12}$ cm	*6.431*

INTRODUCTION

0.1. An atomic nucleus is characterized by two integers, the *charge number* Z, expressing that its total electric charge is a multiple of the electronic charge, and the *mass number* A, according to which its total mass is approximately equal to A times the proton mass.

The interpretation of these properties which first presented itself, was to assume the nucleus to consist of A *protons* and $A-Z$ *electrons*. The phenomenon of β-disintegration, i.e. spontaneous emission of electrons, exhibited by some nuclei, seemed to support this view, although it appeared very difficult to account for the fact that the emitted electrons present a continuous energy distribution, while both the original and the product nucleus are in states of well-defined energy. In fact, the only way to save the principle of energy conservation in this case is to assume that the amount of energy necessary to restore the energy balance of the process is carried away, simultaneously with the emission of the electron, by some agent not reacting in any appreciable way with the environment; for this elusory function a particle of zero charge and zero (or very small) mass has been proposed, and given the name *neutrino*.

However, the simple conception of a proton-electron structure of nuclei proved insufficient in other respects. When a nucleus possesses a sufficiently stable fundamental state, in which it will subsist in atomic or molecular systems, it is possible to determine its *total angular momentum* * in this state, as well as its so-called *statistics*, i.e. the symmetry properties of the wave-function of any system of such nuclei with respect to the sets of coordinates pertaining to each of them. As is well-known, contradictions arose between the observed angular momentum and statistics of several nuclei and those predicted on the proton-electron structure theory. A consistent picture of nuclear constitution, embodying besides charge and mass number also the angular momentum and statistics, could only be obtained after the discovery of the *neutron*, an electrically neutral particle of about the same mass as the proton: assuming the neutron to have the same spin and statistics as the proton, it then became possible, in full agreement with the empirical data on angular momenta and statistics of nuclei, to conceive the nucleus (A, Z) as built up of Z protons and $A-Z$ neutrons. It will often be convenient, on this picture, to treat proton and neutron as two states — differing in charge and to a small extent in mass — of the nuclear constituent particle, which will be called a *nucleon*.

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* The older denomination "nuclear spin" for the total angular momentum of a nucleus is liable to cause confusion and should be discarded. We shall reserve the name *spin* for the intrinsic angular momentum of an *elementary* particle.

In order, now, to account for the β-disintegration phenomena, a mechanism has to be devised, by which a nucleus, with adequate change of its charge number, would emit a (positive or negative) electron together with a neutrino. This can be achieved, in analogy with electromagnetic theory, by postulating a special kind of interaction between a nucleon and a pair of light particles, or *leptons* *, consisting of an electron and a neutrino. Just as the existence of an interaction between charged particles and the electromagnetic field provides for the emission of photons by the charged particles, the interaction between nucleons and "lepton field" will give rise to the emission processes observed as β-decay.

If we now try to get a more quantitative account of the general picture of nuclear constitution just outlined, we are here — in contrast to the case of atomic constitution — left without any theoretical guidance comparable in reliability with the correspondence argument of quantum theory. The law of the forces acting between nucleons (apart from the ordinary Coulomb interaction of protons) cannot be inferred from any unambiguous theoretical reasoning and must ultimately result from the analysis of empirical data. For instance, the difference between the mass of a nucleus and the sum of the masses of the constituting nucleons, its so-called *mass-defect,* gives us directly (owing to the relativistic relation between mass and energy) the total binding energy of these constituents. On the other hand, the study of the *scattering* of nucleons yields valuable information on the law of interaction. With the help of this and other similar experimental evidence it already proves possible to restrict to a considerable extent the arbitrariness in the choice of a suitable law of nuclear force.

But in order to be really fruitful, such an analysis of the empirical material must clearly be combined with a discussion of the diverse possibilities which present themselves from more theoretical points of view. In particular, a theoretical approach to the problem of nuclear forces can be made on the line of the general *field* concept, well-known from the theory of electromagnetic forces. There is a great variety of *a priori* admissible "nuclear fields"; the field equations are conveniently assumed to be of such a form that, according to the fundamental relation between field (wave) and particle of quantum theory, the field may be associated with some kind of particle characterized by a given mass and spin, or with a pair of such particles appearing or disappearing together (like the "lepton field" discussed above in connexion with the theory of β-decay). Among these, a type of field introduced by Yukawa offers a special interest, since the associated particles (when charged) unexpectedly appear to be identical

* Following a suggestion of Prof. C. Møller, I adopt — as a pendant to "nucleon" — the denomination "lepton" (from λεπτός, small, thin, delicate) to denote a particle of small mass, irrespective of its charge; i.e. a lepton would be susceptible to two kinds of states, in which it appears as an electron and a neutrino, respectively. The word "electron" retains its original meaning of a particle of small mass with an elementary charge of either sign. When it is necessary to indicate the sign of the charge, the words "positon" and "negaton" may be used.

with the *mesons* which constitute the penetrating component of cosmic radiation.

0.2 When attempting to set up a general expression for the interaction energy between nucleons, one is faced with a number of questions concerning the dependence of this energy on the various parameters which characterize the state of the interacting particles.

(1) In the first place, one has to consider the dependence on the number of interacting nucleons: besides the usual interactions between pairs of particles, there may arise direct interactions between groups of more than two nucleons. In fact, such *many-body forces* are a direct consequence of any field theory; but on account of the unsatisfactory state of the quantum theory of fields, little can be said at present on the relative importance of the different many-body interactions. It seems, however, that the main features of nuclear constitution can be accounted for on the assumption that all but pair interactions play only a secondary part.

(2) The next question is, whether the nuclear forces may be regarded as predominantly *static*. In this respect, one has to examine their dependence not only on the translational velocities or momenta of the nucleons, but also on the motions of their spins and on the rate of exchange of electric charge between them (as expressed by the time variations of the coordinates characterizing their proton or neutron states). While the effect of the translational velocities can generally be assumed to be small, this is by no means so certain as regards the spin motions and charge variations, which very much depend on the form of field theory adopted. We shall in the following mostly deal with mainly static interactions.

(3) On this last assumption, the dependence of the static potential between two nucleons on the space, spin and charge coordinates has still to be investigated. Besides the distance between the two nucleons, one may expect also that the orientation of the spins with respect to the line joining the nucleons will enter into the expression for the nuclear potential, and it is a vexed question whether this potential is essentially *central* (only distance-dependent) or non-central.

Although we shall in the present work endeavour to follow as far as possible an inductive course of argument, inevitably part of the analysis of experimental evidence will be based on some definite hypothesis on nuclear interaction. We shall then treat most extensively the case of central interaction, which, on account of its greater simplicity, is especially suited to bring out the essential features of the argument. Afterwards, it will be possible to give a more cursory discussion of non-central and non-static interactions.

In this way, the book naturally divides itself into four parts. In the first one is attempted a general survey, of a more qualitative character, of the bearing of the empirical data regarding atomic nuclei on the problem of the nature of nuclear forces. This will immediately yield such features of

universal validity as the *limited range* and *limited binding power* of these forces (the latter property being quite analogous to the well-known *saturation* of the chemical valency bonds). Further, important indications can be obtained on the *charge dependence* of the nuclear interactions: the existence of forces between like nucleons * besides a proton-neutron interaction is disclosed and strong evidence is adduced of the general symmetry of nuclear force (exclusive of Coulomb repulsion between protons) with respect to the charges of the interacting nucleons.

Parts II and III are concerned with a more detailed analysis from the standpoint of central interaction. In Part II it is shown how the evidence of the deuteron and of proton-neutron and proton-proton scattering leads to a partial determination of the central potential. The chief result is the derivation of the so-called *charge independence* property of this potential: to a certain approximation, the proper nuclear interaction turns out to be the same for any pair of nucleons in the same configuration, irrespective of their charges. This fixes the analytical form of the potential as regards the dependence on spin and charge coordinates. The detailed distance dependence cannot be ascertained; but for any given law of force, the range and strength of the potential are more or less accurately fixed in magnitude. The numerical values of two constant parameters, entering into the dependence on spin and charge variables, remain as yet undetermined. They can only be fixed by the consideration of more complex nuclei, to which the following Part is devoted.

Part III, in fact, treats of the more or less adequate pictures of nuclear structure which can be outlined on the basis of central interactions. We have here especially to do with the calculation of the energies of the *stationary states of nuclei*, together with the discussion of other properties of these states needed for that purpose. It is found that the requirement that the nuclear potential exhibit the right saturation properties serves, as already mentioned, to complete the determination of the numerical values of the interaction parameters. To arrive at this result, a rough model of the nucleus suffices; it is then shown on what lines a more refined treatment can be attempted. In particular, properties of systems of three nucleons can be discussed to such extent as to afford an important corroboration of the conclusions drawn from the study of the saturation properties of the nuclear forces.

Finally, we discuss in Part IV the possible contribution of *non-central and non-static couplings* to the nuclear interaction. After a survey of the various possible types of such couplings which might be expected to occur, an analysis of the relevant empirical evidence, chiefly on the proton-neutron system, is carried out with a view to ascertaining in how far the effects concerned might either be attributable to relatively small deviations from central interaction or, on the contrary, reveal the essential inadequacy of the latter. Different theoretical possibilities are tested, but no definitive conclusion is reached.

CONTENTS

PART I

GENERAL FEATURES OF NUCLEAR FORCES

CHAPTER I

PROPERTIES OF ELEMENTARY PARTICLES

1.0. This preliminary chapter reviews in a general way the main properties of the elementary particles of different kinds, which will play a role in the following discussion.

1.1. Leptons

1.11. *Exclusion principle and anti-particles.* Electrons and neutrinos are elementary particles with a spin of half a unit, described by Dirac's well-known wave-equation. It is a consequence of very general features of quantum theory (PAULI [40]) that such particles obey the exclusion principle. This principle states that no two particles may be in the same state, i.e. have the same values for all their coordinates. It follows from this that the wave function of a system of such particles must be antisymmetric in all the sets of coordinates of the single particles; this symmetry property is expressed by the phrase "Fermi statistics". The formulation of the exclusion principle is logically possible only for particles of half-integral spin (but not for particles of integral spin); and it *must* be assumed to hold in order to prevent the occurrence of transitions of such particles to states of negative energy: this is achieved by introducing an infinite number of particles filling up all the states of negative energy, one in each such state. The particles in question are assumed not to produce any field as long as they are in the negative energy states, but to behave normally when they go over to a state of positive energy. When such a transition occurs, a "hole", i.e. an unoccupied state, is left in the distribution of particles of negative energy: such a hole will also give rise to observable effects, which can be attributed to an *anti-particle* of positive energy. Thus a hole in the distribution of negative electrons of negative energy will behave just like a positive electron. The transition of a particle from a state of negative to a state of positive energy is therefore to be interpreted as the creation of a pair consisting of a particle and an anti-particle of the kind considered: the inverse transition into an unoccupied state of negative energy corresponds to the annihilation of such a pair.

1.12. *Types of β-processes.* While electrons and anti-electrons are easily distinguished by the sign of their charge, there is no practical means of distinguishing neutrinos from anti-neutrinos. Since the expression for the interaction energy between a nucleon and a pair of leptons takes a simpler form when one of the leptons is associated with an anti-lepton of

the other kind, the process of β^--decay is usually described as the emission of a negative electron and an anti-neutrino, while the β^+-activity consists of the emission of a positon and a neutrino. The emission of an anti-lepton is here formally treated as the absorption of a lepton of negative energy. Besides the β^+-decay, there is thus another competing process leading to the same kind of product nucleus: it is the absorption of a negaton, accompanied by the emission of a neutrino; as the negaton absorbed will usually be one in the K-shell of the atom, nearest to the nucleus, this phenomenon is called *"K-capture"*.

1.13. *Energy balances of β-processes.* The energy balance of these different types of β-transitions is easily established in terms of the masses of the nuclei involved. The difference of these masses is equal, in the case of a β^-- or β^+-transition, to the sum of the masses of the emitted leptons, plus their total kinetic energy, which is given by the maximum energy of the electron spectrum. From the study of the intensity distribution of this spectrum near the maximum energy, it can be concluded that the mass of the neutrino is very probably zero. Denoting by $M(A, Z)$ the mass of the nucleus of mass number A and charge number Z, by m the rest mass of the electron and by K_β the limiting kinetic energy of the β-spectrum, the energy balance may thus be written

$$M(A, Z) - M(A, Z \pm 1) = m + K_\beta . \tag{1}$$

Likewise, the balance of the K-capture process takes the form

$$M(A, Z) - M(A, Z-1) = -m + K_\nu , \tag{2}$$

K_ν denoting the energy of the emitted neutrino; strictly speaking, one should add to this the binding energy (in absolute value) of the electron in the K-shell, but in comparison with the orders of magnitude involved in nuclear processes, such atomic binding energies are (except for very high atomic numbers) quite negligible quantities.

1.131. *Stability with respect to β-transitions.* It must be observed that except in the region of the heaviest nuclei, where the possibility of emission of α-particles has also to be envisaged, it is just the β-decay that conditions the distribution of stable nuclei along the whole range of mass numbers. Hence the importance of the above energy balances in allowing us to formulate the condition of stability of any nucleus with respect to β-transitions. This condition takes an especially simple form if the energy balances are written in terms of the *atomic weights*, i.e. the masses of the neutral atoms, instead of the masses of the nuclei. If the atomic weights are denoted by $W(A, Z)$, we have:

$$(A, Z) \xrightarrow{\beta^-} (A, Z+1) : \quad W(A, Z) - W(A, Z+1) = K_\beta \tag{3}$$

$$(A, Z) \xrightarrow{\beta^+} (A, Z-1) : \quad W(A, Z) - W(A, Z-1) = 2m + K_\beta \tag{4}$$

$$(A, Z) \xrightarrow{K} (A, Z-1) : \quad W(A, Z) - W(A, Z-1) = K_\nu . \tag{5}$$

We thus see that the most stringent *stability condition* is derived from the consideration of β^--decay and K-capture and can be enunciated as follows: For a nucleus to be stable against β-transitions, its atomic weight must be lower than those of the neighbouring nuclei of the same mass number, but with charge numbers larger or smaller by one.

1.132. *Existence of the neutrino.* Interesting evidence in support of the existence of the neutrino has recently arisen from various sides. CRANE and HALPERN [38, 39] observed the β-decay of $^{38}_{17}Cl$ in the cloud chamber by photographing the tracks a short time after the end of the expansion: the (diffuse) β-ray track yielded, by magnetic deflection, the momentum of the emitted electron, while a measure for that of the recoiling nucleus was obtained by counting the condensation droplets clustering around the spot where the disintegration had taken place. It appears that the conservation laws are not satisfied for the system consisting of β-ray and product nucleus only, the momentum taken up by the nucleus being in *all* cases too large.

Another method of detecting recoil atoms (or rather ions) consists in accelerating them by a controllable electrostatic potential towards some suitable collecting device. This method thus allows in principle of a determination of the energy distribution of the recoil atoms, but it is beset with considerable experimental difficulties. It has been tried for the first time by LEIPUNSKI [36*a*], who investigated the positon emitter ^{11}C; in other forms, it has more recently been used by JACOBSEN and KOFOED HANSEN [45] with ^{88}Kr originating from Uranium fission, and by ALLEN [42] with ^{7}Be. This last instance is interesting inasmuch as one has to do with a K-capture transmutation, so that there is no complication due to emitted electrons and the evidence from the recoil nuclei is very direct indeed. The results of the last two experiments, although far from perfect, are in quite satisfactory agreement with the neutrino hypothesis.

Evidence on the emission of a neutrino in meson decay, which will be mentioned later (*1.332*), has also — when combined with the meson theory of β-activity (*0.1, 1.322*) — a bearing on the question and should accordingly be put on record here.

1.2. Nucleons

1.21. *The proton.* The proton being easily available as a hydrogen nucleus, its properties can be determined experimentally by the same methods as apply to the general investigation of nuclei. Thus it can be derived from the alternation of intensities in the band spectrum of the hydrogen molecule * that protons, like electrons, have a spin $\frac{1}{2}$ and obey the exclusion principle. The conclusion as regards the value of the spin has been confirmed by the method of magnetic deflection of atomic and molecular rays, developed by RABI and his collaborators **, — a method which at the same time yields the value of the magnetic moment.

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* See, for instance, KOPFERMANN [39], Chapt. IV.

** First determinations by RABI *et al.* [36], more refined measurements by RABI *et al.* [39], MILLMAN and KUSCH [41]. A general account of the method has been given by KELLOGG and MILLMAN [46].

In view of these results, it is certainly tempting to assume for the proton a wave-equation of the same type as that established by Dirac for the electron, the only differences being the sign of the charge and the numerical value of the mass. Still, it must be admitted that such a procedure is not free from uncertainties. In fact, as has been pointed out by BOHR [34], Dirac's theory, being based on the idealisation of a point electron, may only be safely applied, for the treatment of proper relativistic effects (such as the spin properties), to particles of dimensions much smaller than the critical radius $\hbar/M$ (M being the mass of the particle) of the domain within which relativity features become preponderant. Now, the electron radius, defined by the condition that outside a region of such dimensions the electromagnetic field created by the electron approximates to that of a point charge, is of the order

$$d = \frac{e^2}{m}, \tag{1}$$

e denoting the elementary charge; the condition just enunciated is then fulfilled owing to the smallness of the fine structure constant $e^2/\hbar$. But the proton radius, determined by its proper nuclear field, turns out to be again of the order of magnitude d, which due to the smallness of the mass-ratio m/M_p (where M_p denotes the proton mass) is actually larger than $\hbar/M_p$.

From this point of view, it might be taken as an indication of the insufficiency of Dirac's theory for the proton that the value of the magnetic moment of the free proton (at rest) which follows from Rabi's measurements, mentioned above, is not that of a *nuclear magneton*

$$\mu_0 = \frac{\hbar e}{2 M_p}, \tag{2}$$

predicted by the theory, but $\mu_p \cdot \mu_0$ with

$$\mu_p = 2{,}7896 \pm 0{,}0008. \tag{3}$$

It is true that this anomalous value of the magnetic moment can in principle be understood when it is realised that any nucleon, as the ultimate source of the lepton field responsible for the phenomena of β-decay, must be surrounded by a certain distribution of electricity. For instance, if we assume the lepton field to be directly produced by the nucleon, this field itself will obviously carry electric charge; we may, on the other hand, be led to assume the nucleon to be the source of some intermediate field, which in its turn gives rise to the lepton field — the meson field, as we shall see, affords an example of such an intermediate agent —; then, the intermediate field will also have to be charged. Provided the production of the field is assumed to depend on the spin of the nucleon, its distribution of electric current will be liable to give rise, on the average, to a magnetic moment in the direction of the spin. Such a magnetic moment

will be inseparably connected with the nucleon and will thus be attributed to it. On this picture, the total magnetic moment of the proton, directed along the spin axis, will in fact turn out to be larger than the normal value of one nuclear magneton by the amount $\mu_p - 1$. But this explanation of the magnetic anomaly makes it clear that a treatment based on Dirac's equation must at any rate be completed by the consideration of the effects of the proper (lepton or meson) field; and it may well be that this modification is nothing more than some convenient approximation to a more adequate description.

1.22. *The neutron.* The properties of neutrons are much more difficult of access than those of protons. Thus no direct determination of neutron spin or statistics can be obtained; but the indirect evidence on these points, resulting from the neutron-proton picture of nuclear constitution and well-established principles of quantum theory, is very strong indeed. In fact, if we assume the neutron also having a spin $\frac{1}{2}$ and obeying the exclusion principle, it follows from general theorems of quantum mechanics that any system of an odd number of nucleons must have a half-integral angular momentum and obey Fermi statistics, while any system of an even number of nucleons has an integral angular momentum and obeys Bose statistics (meaning that the wave function of such a system is symmetrical in the sets of coordinates pertaining to the constituent nucleons); and this rule actually holds good without exception for the numerous nuclei which have been investigated *. In particular, the deuteron, a nucleus consisting of a proton and a neutron, has been found (MURPHY and JOHNSTON [34]) to have an angular momentum 1 and to follow Bose statistics.

A precise value of the neutron mass M_n can be derived ** from a mass-spectrographic comparison (MATTAUCH [38]) of the hydrogen molecule with the deuterium atom, combined with the accurate determination of the binding energy of the deuteron (*6.11*–1). The previously mentioned mass difference is found to correspond to an energy of $1{,}434 \pm 0{,}002$ MeV, while the deuteron binding energy amounts to $2{,}185 \pm 0{,}006$ MeV. If the deuteron mass is denoted by M_d, we therefore have

$$2M_p - M_d + m = 1{,}434$$

$$M_n + M_p - M_d = 2{,}185,$$

whence (since $m = 0{,}51$ MeV)

$$M_n - M_p = 1{,}26 \text{ MeV} \tag{4}$$

or

$$M_n - M_p = 2{,}47\, m. \tag{5}$$

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* For the spins, see *A2.21*. For the statistics, see KOPFERMANN [39], p. 226.

** A critical survey of the complete evidence available on this point is given by STEPHENS [47].

In consequence of this large value of its mass, the neutron must be expected to be β-active. In fact, its spontaneous β-decay is energetically possible even if the mass of the neutrino should be as large as that of the electron. Applying to this case the general theory of β-disintegration of light nuclei, the life-time of the neutron can be estimated (*A1.211*) at about half an hour. This suffices to make it clear that the β-decay of the free neutron cannot be observed under the ordinary experimental conditions of production and detection of these particles *.

The fact that the neutron possesses a magnetic moment is perhaps even stranger, in view of the absence of electric charge, than the anomalous value of the proton magnetic moment. It can, however, be accounted for exactly in the same way, at least in principle. It may even be seen that the simplest assumptions as regards the interaction between a nucleon and its charged field lead, in a first approximation, to magnetic moments of the same absolute value, but opposite signs, for a proton and a neutron state not differing in other respects. The neutron will therefore be expected to exhibit a magnetic moment in the direction opposite to that of the spin, and approximately equal in absolute value to $\mu_p - 1$ nuclear magnetons. The first direct measurement, performed by ALVAREZ and BLOCH [40*a*], gave for the neutron magnetic moment $\mu_n = -1{,}935 \pm 0{,}02$ nuclear magnetons, the minus sign indicating that its direction is opposite to that of the spin. This value is now superseded by the much more accurate one,

$$\mu_n = -1{,}9103 \pm 0{,}0012, \tag{6}$$

obtained in the Argonne laboratory (ARNOLD and ROBERTS [46]). Comparing with (3), we find so far a satisfactory agreement with the theoretical predictions; the remaining small asymmetry can be ascribed to higher order effects. Unfortunately, in the present state of quantum field theory, it is impossible to give a more quantitative treatment of the whole effect, leading to a determination of the numerical values of μ_p and μ_n, without making use of some hazardous procedure to secure a convergent result.

1.23. *The anti-proton.* If we are to take the application of Dirac's theory to nucleons seriously, we have to accept the existence of *anti-nucleons*, in particular of *anti-protons*, i.e. protons of negative charge. Of course, anti-protons cannot co-exist with ordinary protons in stable configurations, owing to the tendency of proton pairs to annihilation, e.g. with emission of light quanta (like electrons). But they might penetrate the earth's atmosphere as a part of cosmic radiation or be produced in the atmosphere by other energetic particles of this radiation.

The production of a pair of protons in any process involving electromagnetic interactions will have a probability roughly $(m/M_p)^2 \approx 3 \cdot 10^{-7}$

* Negative experimental evidence on this point is afforded by GILBERT, SMITH and FREMLIN [37].

times smaller than the analogous production of an electron pair (of the same velocities); but there are other possibilities of proton pair production due to the nuclear field, especially if the latter is identified with the meson field observed in cosmic radiation (*0.1*). From a careful discussion by McCONNELL [45] * it results that the cross-section for proton pair production by fast mesons, although of a larger order of magnitude than those of electromagnetic processes, always remains very small indeed. On the other hand, the annihilation in matter of anti-protons with emission of mesons or photons has also a quite small probability, unless the annihilating protons are very slow; so that more prolonged observations than hitherto performed of the proton component of cosmic radiation at low altitude should at length reveal the existence of anti-protons produced at higher levels. However, the difficulty of detecting these anti-protons is considerably increased by the occurrence of another process of relatively large probability, viz. that by which a proton, — negative as well as positive —, in traversing matter, loses its charge by emitting a meson. It is interesting to note that for very slow protons and anti-protons, the annihilation to the form of photons will become predominant; the resulting life-time of an anti-proton at rest being estimated at about 10^{-4} sec.

As to the occurrence of anti-protons in the primary cosmic radiation, a case has been presented by ARLEY [45, 46*a*, 46*b*] in favour of this eventuality. By annihilation in the atmosphere, the anti-protons would ultimately give rise to the soft component of the cosmic radiation, while the ordinary protons are the origin of the hard component. Again, it should be possible to observe at sea level some of the primary anti-protons. However, ARLEY does not pay sufficient attention to the process of meson production by the negative protons, the consideration of which would seem to upset his argument **.

The primary cosmic radiation itself has recently been the subject of an interesting hypothesis, put forward by KLEIN [45] ***. He observes that part of the galaxies may well consist of "reversed matter", i.e. of atoms built up of anti-nucleons and anti-electrons; such atoms could in any case not be distinguished from ordinary ones by spectroscopic properties. In the intergalactic regions, collisions may then occur between ordinary and reversed atoms, strayed from their respective galaxies. If the two atoms have the same number of constituent particles, their complete annihilation will result in a radiation composed of mesons, photons and leptons; if they have different constitutions, part of the annihilation energy will also liberate nucleons and anti-nucleons (the neutrons and anti-neutrons soon transforming, owing to their β-radioactivity, into protons and anti-protons). In this way, the composition of the primary cosmic radiation would at least

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* See further McCONNELL and JÁNOSSY [47].

** This has kindly been pointed out to me by Prof. HEITLER.

*** See also ARLEY's [45] criticism.

qualitatively (also as regards the order of magnitude of the energies of the different kinds of particles) be accounted for.

However this may be, we have, in the treatment of nuclear systems, only to do with an approximation involving at most effects of the first order in the nucleon velocities *; in these cases, the uncertainties connected with the use of Dirac's equation in the proper relativistic domain do not occur.

1.231. *The anti-neutron.* At first sight, it might be thought that the anti-neutron is undistinguishable from the ordinary neutron. But it must not be forgotten that its proper nuclear field manifests itself by a magnetic moment, which (for the simple forms of nuclear fields usually adopted) turns out to have opposite signs for neutron and anti-neutron. The theory thus provides for the creation and annihilation of pairs of neutrons of opposite magnetic moments through the interaction with the nuclear (meson) field.

1.3. Mesons

1.31. *The field concept.* The use of the field concept to account for the interaction between particles will most conveniently be explained with the help of the classical example of electrodynamics. It is well-known how, according to Faraday's and Maxwell's ideas, forces acting between electric charges or the Ampère force between elements of current, are derived from a process of transmission of force by means of an electromagnetic *field*, produced by the charges or elements of current and acting on them. A convenient formal representation of this process is afforded by the electromagnetic *potential*. This is a four-vector A_i, the components of which satisfy the differential equations

$$\Box A_i = -4\pi s_i, \tag{1}$$

where s_i denotes the four-vector density of current and charge. Limiting ourselves to the stationary case, we have to do with a set of Poisson equations

$$\triangle A_i = -4\pi s_i, \tag{2}$$

and we can express the solutions we want with the help of the corresponding *Green's function*. This function is a solution φ_0 of the homogeneous equation

$$\triangle \varphi_0 = 0 \tag{3}$$

with a point singularity, i.e. representing physically the potential of a point source. In our case (boundary condition: potential vanishing at

* By this abbreviated expression we mean, here and in the following, the order of magnitude of the ratio between the mean value of the velocities of nucleons in the system considered and the velocity of light.

infinity) we have simply

$$\varphi_0(P';P)=\frac{1}{r}, \tag{4}$$

if r denotes the distance between the point P and the point P' where the singularity is situated. The solutions of the Poisson equations (2) are then given by

$$A_l(P)=\int s_l(P')\,\varphi_0(P';P)\,dv_{P'}, \tag{5}$$

expressing how each volume-element in which a source of strength $s_l(P')dv_{P'}$ is located yields an additive contribution to the potential at point P.

Now, the interaction energy between the sources and the field defined by the potential is given by

$$\mathcal{V}_{\text{el. magn.}}=-\tfrac{1}{2}\int A_l(P)\,s^l(P)\,dv_P. \tag{6}$$

Eliminating the potential, we get

$$\mathcal{V}_{\text{el. magn}}=-\tfrac{1}{2}\int s_l(P')\,dv_{P'}\,\varphi_0(P';P)\,dv_P\,s^l(P), \tag{7}$$

i.e. just the usual expression of the potential energy of the above-mentioned forces acting directly between distant sources. The field has served as an intermediary to connect the sources at different points P, P' and has then disappeared from the final result.

1.32. *The meson field.* As we shall see presently, a fundamental property of the nuclear forces, contrasting with those of electromagnetic interactions, is their *limited range*. The representation of short range forces by means of a field can of course be achieved in many ways, but one of the simplest is the following. Instead of the Laplace equation (3) for the determination of Green's function, let us consider the equation

$$\Delta\varphi-\varkappa^2\varphi=0; \tag{8}$$

the corresponding Green's function is

$$\varphi=\frac{1}{r}\,e^{-\varkappa r}, \tag{9}$$

which is just suited to represent a force not extending appreciably beyond distances of the order $\varkappa^{-1}$. Now, the differential equations for the potential components of the most general field of this type are easily set up: they will clearly be of the form

$$\Box\psi-\varkappa^2\psi=-4\pi s. \tag{10}$$

Here the covariance properties of the potential ψ and the source density s are left entirely open; an equation of the type (10) must be understood to

hold separately for each component of ψ and the corresponding component of s.

The remarkable property of this particular form of field of finite range is that the homogeneous field equations

$$\Box\psi - \varkappa^2\psi = 0, \tag{11}$$

which describe the pure field in absence of any source, are formally identical with the wave-equations of a particle of mass M_m, this last quantity being connected with the range constant $\varkappa$ by the relation

$$\varkappa = \frac{M_m}{\hbar}. \tag{12}$$

In other words, the field of limited range here considered is associated, according to the principles of quantum mechanics, with a certain kind of particle, whose mass is related to the range of the field by equation (12). This is the essential point stressed by YUKAWA [35].

The range of the nuclear forces being of the same order of magnitude as the fundamental length d, defined by (*1.21*–1), the mass of the particles associated with the nuclear field should, by (12), be of the order of $\hbar/e^2 \approx 137$ times that of the electron, thus intermediate between the masses of electron and nucleon: hence the name *meson* proposed for such particles, and the corresponding denomination of "meson field" for the wave aspect defined by equations (11). The covariance properties of the meson field determine the intrinsic angular momentum, or spin, of the meson; apart from a spinor field (like that of the electron), corresponding to a spin $\frac{1}{2}$, the simplest possibilities are the scalar and pseudoscalar * fields of spin 0 and the vector and pseudovector fields of spin 1.

1.321. *The mass-range relation.* In consideration of its importance, it will be worth while discussing Yukawa's relation (12) between particle mass and field range from a less formal point of view. It will appear that this universal relation is deeply rooted in the fundamental principles of quantum theory. For this purpose it will be convenient to introduce the concept of *virtual transition*. This we can again do by starting from the case of electrodynamics.

The particles associated with the electromagnetic field are the *photons*. We can try to describe the mechanism of transmission of force by the electromagnetic field in corpuscular language, by making use of the photon concept. We should then like to say that the sources located in one definite volume-element emit photons, which are subsequently absorbed by the charges and currents contained in another volume-element and in this

* The prefix *pseudo* indicates a difference of behaviour of the quantity considered under symmetry transformations (reflexions) with respect to a point (or plane). While a scalar remains invariant under such transformations, a pseudoscalar changes sign; more generally, a pseudotensor transforms like the product of a tensor and a pseudoscalar.

way transport momentum and energy from the first volume-element to the second. But we must be very careful with such a picture. There can be no question of *actual* emission and absorption processes, because such processes are in this case not permitted by the precision with which the energies and times involved can be defined. Indeed, in order to disclose an actual transition of a photon between two volume-elements at a distance r from each other, we should need to be able to capture such a photon on its way and to identify it as such, i.e. to attribute to it a definite frequency. However, this is impossible: since the time at our disposal for the capture and identification of the photon is at most r/c, we cannot define its energy better than with a latitude ΔE given, on account of the uncertainty relation $\Delta E \cdot \Delta(ct) \approx \hbar$, by

$$\Delta E \approx \hbar/r,$$

which means for the (circular) wave-number ν an uncertainty

$$\Delta\nu \approx 1/r.$$

If, nevertheless, we wish to use the corpuscular language, we must therefore not lose sight of the symbolical character of the imagined transport of energy by photons; and of this we are reminded by the phrase "*virtual* photon transition". Such virtual transitions are to be sharply distinguished from the actual ones; the latter must always, in contrast to the former, be compatible with the law of energy conservation.

Let us now consider the transmission of force by a field associated with particles of mass M_m. We can describe this transmission process, in the corpuscular language, as a sequence of virtual emissions and absorptions of particles by the sources. But there is a condition to be fulfilled for the consistency of this picture. We have to express the fact that it must be impossible to identify the particle on its way in the time between the emission and the absorption. This time is here r/v (v being the velocity of the particle) and the corresponding uncertainty in the definition of the particle energy, $\hbar v/rc$, must be larger than M_m:

$$\hbar v/rc > M_m,$$

or, since $v \leq c$,

$$r < \frac{\hbar}{M_m}.$$

The meaning of this inequality is that our transmission mechanism cannot extend beyond a distance $\hbar/M_m$. The relation between range of the force and mass of the particle is thus seen to be a rather direct consequence of the fundamental uncertainty relation for energy and time *.

We here meet — as emphasized by the concept of virtual transition — with a "non-visualizable" feature of quantum theory of exactly the same

* This argument is due to WICK [38].

kind as the fact that no mechanical orbit can be attributed to an electron in a stationary state of an atom. And this particular instance just affords a striking confirmation of the contention that the limitation of applicability of classical concepts, expressed by the uncertainty relations, in reality gives us a greater freedom to introduce new conceptions, which would not be possible from the classical point of view. We see, in particular, what a considerable extension the field idea gets in quantum theory, where every kind of particle is automatically associated with a field and thereby provides new possibilities of transmission of force, i.e. of interaction *.

1.322. *Interaction of meson field with leptons.* The description of nuclear interaction provided by the meson field can, as pointed out by YUKAWA [35], be completed so as to include the various kinds of β-processes. The meson field will in fact transmit an interaction also between nucleons and leptons, provided a direct coupling of this field with the leptons is introduced. The latter coupling can be chosen in such a way that the interaction between nucleons and leptons to which it gives rise will imply the occurrence of the β-transitions. The emission of a lepton pair by a nucleon, for instance, is on this view conceived as a virtual emission of a charged meson by the nucleon, followed by the virtual transmutation of the meson into a lepton pair.

It must now be observed that besides the virtual processes involved in the interpretation of nuclear forces and β-decay, the meson field theory predicts the occurrence of the corresponding actual processes as soon as the energy eventually required is available. If this theory has any physical reality, we should e.g. expect mesons to be emitted by nuclei in states of sufficient excitation. The interaction between mesons and leptons further implies that such mesons should spontaneously disintegrate into a pair of leptons, consisting of an electron and a neutrino.

1.33. *The mesons of cosmic radiation.* It was therefore a triumph of Yukawa's theory when charged particles of intermediate mass were actually discovered in cosmic radiation. The remarkable agreement between the general properties of these particles and those required in meson theory was further enhanced by conclusive evidence as to the spontaneous decay of the cosmic particles. It is true that as soon as we try to go into a more detailed specification of meson properties, especially as regards their charge and spin, in order to account for a larger number of peculiarities of nuclear structure, the apparent simplicity of the meson field conception

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* The application of the extended field conception of quantum theory to types of quantum interactions already known, such as the chemical valency forces and the peculiar interactions connected with creation and annihilation of electron pairs, has been discussed by HUND [39, 40] (see also BLASS [44]). In this connexion it must be observed that in contrast with the case of stationary fields, discussed here, non-stationary fields will have longer ranges than $\varkappa^{-1}$. It is easy to see that a field of angular frequency $c\nu$ has a range $(\varkappa^2 - \nu^2)^{-1/2}$: an example is given by the chemical forces (HUND [39]).

disappears and it becomes increasingly difficult to secure complete harmony with the empirical data derived from cosmic radiation. Still, from the point of view of field theory, as outlined above, we may state that the new cosmic particles must at least partly be responsible, through their field properties, for the nuclear interactions and β-processes, and in this sense we are fully entitled to identify them with the mesons theoretically introduced by Yukawa. We must of course be prepared for the eventuality that other kinds of mesons, which for some reason escape observation under present conditions, may also play a part, perhaps even the main part, in bringing about the interactions which govern the behaviour of nucleons and leptons.

Leaving such matters for later discussion, we shall now, from the rather scanty evidence on cosmic ray mesons, only discuss the most firmly established features. We shall be concerned exclusively with *charged mesons*; charges of both signs being about equally represented *.

1.331. *The meson mass.* The determination of the meson mass from cloud chamber photographs of meson tracks curved in a magnetic field is a precarious affair, owing both to the errors of difficult estimation to which the measurement of the curvature of the track is subject **, and to the large uncertainties affecting the different methods of determination of the velocity of the particle. The latter quantity may be derived from an estimate of the ion density in the track, from a measurement of the range of the meson or of its velocity loss on traversing a dense screen in the cloud chamber, or finally from the analysis of an elastic collision of a meson with an atom, resulting in the expulsion of an electron from the atom. A critical discussion of these various methods (GORODETZKY [42, 46], RICHARD–FOY [45], HUGHES [46]) reveals that the best precision to be hoped for without spending an undue amount of time in waiting for a larger number of suitable photographs to accumulate would not exceed 10 % and that the error of the mass determination is in most cases far greater.

It is therefore not surprising that the results so far obtained exhibit a very wide spread. In fact, critical surveys of older measurements, carried out by WHEELER and LADENBURG [41c] and by BETHE [46], as well as the latest measurements by HUGHES [46] and FRETTER [46], all lend strong support to the conclusion that this spread is not due to an actual spectrum of meson masses, but can be wholly ascribed to experimental uncertainties. As regards the probable value of the unique meson mass, estimates differ somewhat: Bethe favours a value of 200 m; for 26 determinations spread between 142 m and 264 m, Fretter indicates a

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* The evidence establishing that cosmic ray mesons carry one elementary charge is discussed by GORODETZKY [45].

** BETHE [46] has especially emphasized the large influence of multiple scattering on the curvature of the track.

weighted average of 202 m, while 10 tracks obtained by Hughes yield an average of 229 m, with a spread extending from 140 to 340 m. One of the most reliable determinations seems to be that based on a remarkable photograph obtained by Leprince–Ringuet and his collaborators (plate I), and showing both an elastic collision with an atom and the energy loss on traversing a lead plate; the discussion of this case leads to a value

$$M_m = (240 \pm 40)\, m, \tag{13}$$

which, as long as no definite evidence to the contrary is forthcoming, we may assume as the unique mass value of all charged mesons *observed at sea level.*

LEPRINCE–RINGUET and LHÉRITIER [46*a*] have obtained an elastic collision photograph, taken at 1000 m altitude, from which a particle mass of $(990 \pm 120)\, m$ is deduced *. They later [46*b*] found some indirect confirmation of this result in four cloud chamber tracks of abnormal ionizing power, indicating a mass intermediate between those of the proton and the "ordinary" meson. Such "heavy mesons", if they exist, would give rise to nuclear interactions of correspondingly short range, which would probably not be disclosed by the evidence at present available.

1.332. *The meson decay.* Owing to their finite life-time, the mesons observed at sea level cannot be part of the primary cosmic radiation. But the minimum energy, of about 120 MeV, required for the creation of such a meson, is of the order of magnitude met with in the primary radiation, and there is no difficulty in accounting for the production of the observed mesons in the atmosphere. The life-time of the mesons reaching sea-level can be accurately measured by a remarkable method, utilizing to the extreme limit of their efficiency the present resources of experimental technique.

By a system of coincidence and anti-coincidence registrations of counters placed above and below a block of light material, the behaviour of *slow* mesons, penetrating into this "absorber" but not traversing it entirely, can be studied. A series of counters disposed along the sides of the absorber can record any electron emitted in the course of an eventual decay of the meson, following its entry into the absorber. It is now possible to connect the side counters with the top- and bottom counters in such a way that "retarded coincidences" between the arrival of the slow meson and the emission of the decay electron are recorded; i.e., in order to be recorded, the two events mentioned must be separated by a time interval not larger than a certain time, which can be fixed with great accuracy. From measurements performed for different values of this retardation time (of the order of the microsecond), a decay curve can clearly be derived, giving the mean life-time of the slow mesons. This method has been developed independently by RASETTI [41] and by MAZE, CHAMINADE and FRÉON [45]; it has also been applied by CONVERSI and PICCIONI [44, 46*a*]. A still

* BETHE [46] argues, however, that this particle could be a proton.

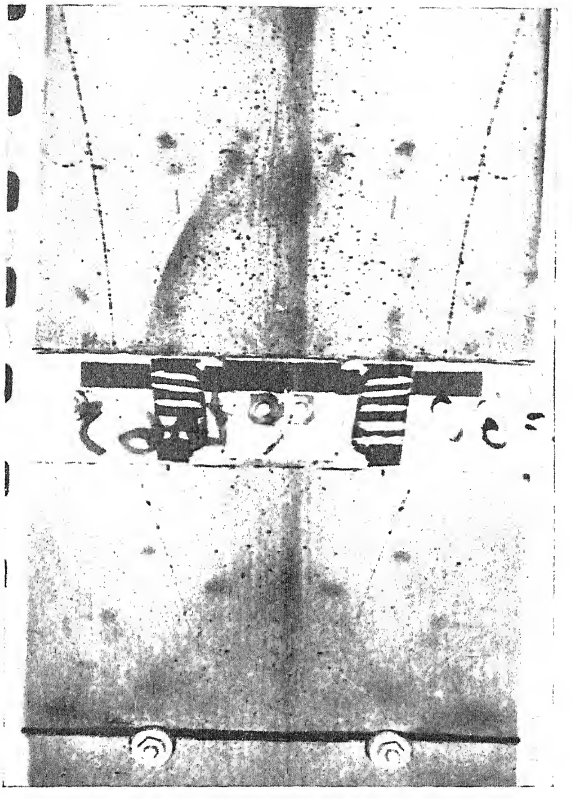

Fig. **1.331.** Cloud chamber track of a meson experiencing an elastic collision in the gas and an energy loss in a lead plate (LEPRINCE–RINGUET *et al.*; see GORODETZKY [42]).

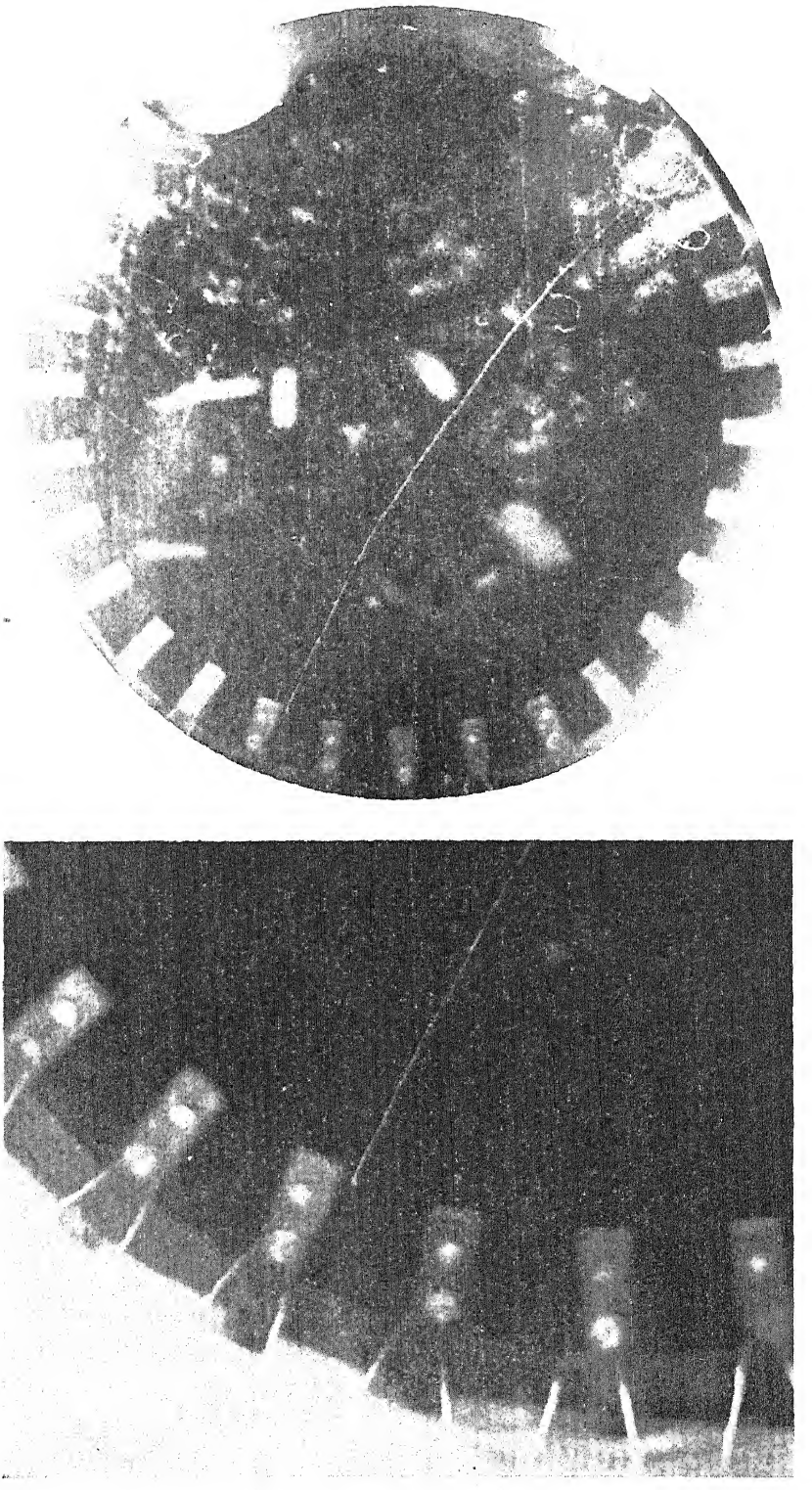

Fig. 1.332. Cloud chamber photograph of a meson decaying with emission of an electron (WILLIAMS and ROBERTS [40a]).

more refined variant of it is due to NERESON and ROSSI [42, 43]: in their apparatus, a special device permits measuring in each case the time-interval separating the recorded events. In this way, a large number of points on the decay curve is obtained and the accuracy of the life-time determination accordingly increased. The results of these difficult measurements are remarkably concordant; the mean life-time of a charged meson at rest (of the kind observed at sea level) is $(2{,}15 \pm 0{,}07) \cdot 10^{-6}$ sec.

An interesting feature of these experiments is that only about one half of the mesons stopped in the absorber are found to decay with emission of an electron (see especially CONVERSI and PICCIONI [46*b*]). This result is interpreted by noting that negatively charged mesons have a larger probability of being captured by nuclei in the absorber. However, further investigation of this aspect of the question has quite recently brought to light a very puzzling phenomenon. Studying the behaviour of slow mesons of both signs, separated by passage through magnetized iron plates, CONVERSI, PANCINI and PICCIONI [47] could detect no decay of the negative mesons when they were stopped in an iron absorber, but did record a production of a number of electrons nearly equal to that of the mesons slowed down, when the absorbing material was carbon. Preliminary experiments of SIGURGEIRSSON and YAMAKAWA [47] seem to disclose a similar anomaly for Be in contrast to a "normal" behaviour of S. According to a suggestion of BOHR, this effect would be due to a quite unusual mechanism of capture of the slow negative meson by atoms of small charge number: there would in fact be a large probability of its being retained on an "orbit" of *large* angular momentum (and consequently with small chance of capture by the nucleus) during a time sufficiently long to permit its decay into leptons.

An immediate consequence of the finite life of the meson is the dependence of the rate of absorption in the atmosphere of the hard component of cosmic radiation on the actual distance of air traversed. On this property, from which the first experimental evidence of meson decay was derived, various methods of estimating the order of magnitude of the meson life-time are based. Since we deal here with mesons of given momentum, the quantity which can be inferred from all such measurements is not the life-time t_0 of the meson at rest, but the quotient of this constant by the meson mass. Critical discussions of the relevant evidence have been given by FRÉON [45] and OPECHOWSKI [43*a*,*b*]; they must be supplemented by the recent work of COCCONI and TONGIORGI [44, 46], which would seem to dispell any doubt as to the constancy of t_0/M_m when the distance traversed by the mesons varies. This means that so far as this evidence goes, it is concerned with only one kind of mesons, for which, according to the last named authors,

$$t_0/M_m = (2{,}6 \pm 0{,}3) \cdot 10^{-6}\,\text{sec}/100\,\text{MeV}.$$

Such a result is in fair agreement with the direct measurements of t_0 and M_m. It is interesting to note that the indirect life-time determinations have furnished the first direct verification of the relativistic law of time dilatation.

Cloud chamber photographs of decaying mesons, first obtained by WILLIAMS *et al.* [40a,b] * (plate I), afford very strong evidence for the emission of a neutrino together with the electron. It should be emphasized, however, that this evidence is as yet only of a qualitative character; it has not been proved that the energy and momentum balance can always be restored by the assumption of the emission of a particle with specified properties. For this purpose, a more systematic study is required.

1.333. *The meson spin.* It has been possible to derive the value of the spin of the mesons observed at sea level from the study of the cascade showers produced by such mesons in a layer of lead. Mesons can initiate showers either by accelerating an electron in a collision or by emitting photons ("Bremsstrahlung") in passing near a nucleus. The yield of these processes turns out (CHRISTY and KUSAKA [41], KUSAKA [43]) to be quite sensitive to the value of the meson spin. Mesons of spin 1 would, according to the theory, produce many more soft secondaries than actually found in experiment. The evidence (LAPP [43, 46], GEORGE [46]) definitely points to the value 0 for the spin of sea-level mesons.

It must, however, be mentioned that neither the measurements nor the theoretical calculations have as yet attained such a degree of accuracy as to exclude completely the possibility of a spin $\frac{1}{2}$. If the mesons had spin $\frac{1}{2}$, they could transmit interactions between nucleons without violating conservation of angular momentum only if they were emitted and absorbed in pairs. Pair field theories, however, have not been successful in accounting for detailed features of nuclear forces; they will not be considered in this book.

1.34. *Many-body forces.* It is clear that the mechanism of transmission of forces provided by the meson field will not only lead, as explained above, to an action at a distance between a pair of particles, but also to such direct interactions between groups of more than two particles. Take, for example, a system of three nucleons. In first approximation the interaction energy will be the sum of the interactions between the nucleon pairs (1)–(2), (2)–(3), (3)–(1). These interactions involve virtual processes of the type: nucleon (1) emits (virtually) a meson, which is absorbed by nucleon (2). The next approximation will first of all bring corrections to these pair interactions, arising from virtual processes such as: nucleon (1) emits two mesons which are both absorbed by nucleon (2). But in addition, it will also involve terms corresponding to the following kinds of virtual processes:

(*a*) nucleon (1) emits two mesons, the one of which is absorbed by nucleon (2), the other by nucleon (3);

(*b*) nucleon (2) and nucleon (3) each emit one meson; both mesons are absorbed by nucleon (1).

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* Another example has been published by SHUTT *et al.* [42].

Such terms will be typical three-body interactions, depending on the mutual distances (1)–(2) and (1)–(3).

A detailed discussion of these many-body interactions unfortunately lies outside the range of well-established validity of present field theory: like other effects of higher approximation they cannot be unambiguously separated from the divergencies occurring in these higher approximations. Nevertheless, if one disregards this fundamental objection, it is possible, on meson theory, to set up the analytical expression of the three-body interactions and to estimate their order of magnitude in stationary states of nuclear systems (PRIMAKOFF and HOLSTEIN [39], JÁNOSSY [39]); compared with the static pair interactions they are found to be at most of the first order in the nucleon velocities. While showing that the many-body forces are in general of secondary importance, this result at the same time warns us that we should not lose sight of them when discussing finer details of nuclear systems of more than two nucleons.

1.35. *Excited states of nucleons.* As already explained, a nucleon is always inseparably bound with its proper meson field, which may carry a certain quantized amount of angular momentum or electric charge, with a correspondingly large amount of energy. This means that the nucleon is in principle susceptible of a whole set of stationary states of different excitation, in which it exhibits values of spin and charge differing from those of its normal or ground state. The possibility of spontaneous emission of mesons from such a state sets an upper limit to the possible number of stable excited states. Naturally, the minimum excitation energy, or, more generally, the energy difference between successive excited states will be the smaller, the more strongly the meson field is coupled with the nucleon. In the case of very strong coupling, therefore, the excitation energies being much smaller than the rest energy of the nucleon, the excited states of this nucleon will appear as *isobars* of the ground state, with anomalous values of the spin and charge. In the opposite limit of very weak coupling, the minimum excitation energy will eventually be so large that the occurrence of excited states of the nucleons could only make itself felt in processes involving much higher energies than those hitherto studied. The nucleons may then for all practical purposes be treated as particles with just the normal, invariable values of spin and charge.

From the magnitude of the nuclear forces, it is immediately apparent that the nuclear field is at any rate much more strongly coupled to its sources than the electromagnetic field, so that the perturbation method developed in connexion with the latter cannot be applied to this case with the same confidence. Accordingly, special procedures have been developed to attack the problem from the strong coupling side; a rigorous treatment based on this assumption is, however, just as impossible as the ordinary one using perturbation theory, on account of the difficulties arising from the idealization of point sources. One is therefore obliged to introduce a

model of extended nucleon, which is necessarily unrelativistic and to a large extent arbitrary. In the same way as, in the theory of the extended electron, the field energy contributes to the inertia of the particle, the proper nuclear field gives rise to "moments of inertia" governing the rate of change of spin and charge. In fact, the excitation energy of the nucleon has just the form of a kinetic energy of rotation; the spin and charge variations can be compared to a "gyroscopic motion", the exact type of which depends on the specific assumptions made on the kinds of meson fields created by the nucleon.

Let us choose some arbitrary function $D(P)$, with $\int D(P)dv = 1$, to denote the spatial distribution of the sources of meson fields of the nucleon, and define by

$$\frac{1}{a} = \int D(P)\,\frac{1}{r}\,D(P')\,dv\,dv' \tag{14}$$

the "radius" a of the nucleon; it must, of course, be assumed to be much smaller than the range of the meson field:

$$a\varkappa \ll 1. \tag{15}$$

If g represents the constant (or some average of the different constants), analogous to the elementary electric charge, which determines the intensity of the coupling of the meson field with the nucleons, the moment of inertia of spin and charge is of the order of magnitude

$$I = \frac{g^2}{a}\cdot\left(\frac{1}{\varkappa}\right)^2, \tag{16}$$

corresponding to a "radius of gyration" of the order of the range and a "mass" defined by the average self-energy of the proper meson field; with the expression (9) for the potential energy and the definition (14), this self-energy is indeed $\approx g^2/a$ on account of (15). The minimum excitation energy will be of the order $\hbar^2/I$ and there will be isobars stable against emission of mesons as long as this quantity is smaller than M_m; using (12) and (16), this gives for the dimensionless constant $g^2/\hbar$, analogous to the fine structure constant of electrodynamics, the critical value $a\varkappa$, such that the condition for strong coupling is

$$\frac{g^2}{\hbar} \gg a\varkappa. \tag{17}$$

Now, the roughest comparison of orders of magnitude of binding energies in atomic and nuclear systems suffices to show that $g^2/\hbar$ must be expected to be rather more than 10 times larger than the corresponding $e^2/\hbar$. Since the parameter a is not restricted otherwise than by the inequality (15), it is therefore not clear from condition (17) whether with such a value of $g^2/\hbar$ we have actually to do with a case of strong coupling, or whether we should rather describe it as one of "intermediate coupling". In

other words, it is not certain from general considerations which one of the two opposite lines of approach, characterized by the assumption of strong or weak coupling, will prove better adapted to the actual situation. Ultimately, of course, a decision as to the existence of (stable or unstable) excited states of nucleons might be reached by direct experiments in the energy region in which these states may be expected to have a marked effect on various phenomena. But even at the present stage, quite definite indications in this respect can be derived from certain consequences of the strong coupling theory concerning properties of nucleons or nuclear systems.

It must be emphasized that even in its ground state, in which its angular momentum is $\frac{1}{2}$ and its charge 0 or 1, the nucleon has very different properties according as the coupling with its proper field is treated as strong or weak. For in the first case the degrees of freedom corresponding to spin and charge, which on weak coupling are quite independent from each other, can no longer be separated, so that we cannot associate a definite spin orientation with a definite value of the charge. The result is that the magnetic moment of the "bare" proton or neutron (i.e. without the contribution due to the current distribution of the proper meson field) no longer has the "normal" value μ_0 or 0: instead, its expectation value becomes entirely symmetrical with respect to the proton and neutron states. Since the same is true of the contribution from the proper nuclear field to the total magnetic moment, the latter quantity turns out to have exactly equal and opposite values for proton and neutron.

This is not the only prediction of the strong coupling theory which is flatly contradicted by experiment. In particular, the properties of the deuteron prove irreconcilable with the assumption of a large spin inertia of the nucleon. We are therefore amply justified in consistently adopting for all theoretical considerations in this book the point of view of *weak coupling theory*. This implies above all that we completely disregard the possible existence of excited states of the nucleons or of any coupling between spin and charge in the normal state, but keep to the usual conception as summarized in the preceding section (*1.2*). However, in view of the numerous and far-reaching modifications which would be necessitated by any departure from these assumptions, a brief account of the main lines of argument of strong coupling theory will be given by Appendix *A*3.

CHAPTER II

NUCLEAR RADII AND MASS-DEFECTS

2.0. The general character of the law of nuclear force can be inferred from the remarkable regularities exhibited by the *nuclear radii* on the one hand, and the *mass-defects* of the stable nuclei on the other. Of course, we are dealing here with strongly schematized features intended only to represent a kind of average behaviour, from which individual nuclei will deviate to some extent. The resulting information on nuclear interactions will accordingly be of qualitative, or at most roughly quantitative character.

2.1. Nuclear radius and range of nuclear forces

Already in 1911 Rutherford's celebrated experiments on the scattering of α-rays by matter revealed in a striking manner that nuclear forces, in contrast to the electrostatic ones, are of very *limited range*. It was in fact observed that the scattering followed the law derived from the assumption of simple Coulomb repulsion between α-ray and nucleus up to closest distances of approach of the order of 10^{-12} cm (for heavy nuclei). The *nuclear radius* could be defined as the critical distance from the centre at which the law of interaction with the α-particle ceases to be the electrostatic repulsion, but begins to vary rapidly with distance, the structure of the nucleus coming into play (see RUTHERFORD *et al.* [30], § 45).

More generally, any process involving a close interaction between a nucleus and a charged particle, such as a proton or an α-particle, will essentially depend on the nuclear radius. For in all such processes, the charged particle, in order to penetrate into the nucleus or to escape from it, must traverse the barrier of potential formed by the Coulomb repulsion acting at large distances and the average attraction exerted by the nuclear forces on the particle at distances smaller than the nuclear radius. The probabilities of occurrence of the processes in question will involve as a factor the transmission coefficient or *penetrability* of the barrier, which will depend, not only on the energy and charge of the particle considered, but also on the nuclear radius.

From a study of such interactions of nuclei with charged particles numerous estimates of nuclear radii can therefore be obtained. For nuclei of the radioactive families, we dispose of the data of the natural α-decay; for lighter nuclei, we can use not only the experiments on scattering of protons or α-particles of high energy, but also those on the disintegration of such nuclei by charged particles. All the evidence (see M & F, p. 58 sq., p. 72 sq., and also PRESENT [41]) is found to be compatible with the simple assumption of an approximately *constant density of "nuclear matter"*

for all nuclei. In other words, the nuclear radius is approximately proportional to the cubic root of the mass number:

$$R = r_0 A^{\frac{1}{3}}, \tag{1}$$

the *"nucleon radius"* r_0 being numerically given by

$$r_0 = 1{,}42 \cdot 10^{-13} \text{ cm}. \tag{2}$$

The significance of this remarkable result will presently (*2.22*) appear when it will be combined with the limited binding power of the nuclear forces.

Another possibility of estimating nuclear radii of light nuclei, which leads to results entirely in accordance with the above, is based on the consideration of β^+-active nuclei with neutron excess —1. It will be discussed later (*3.3*) in another connexion.

2.2. Mass-defects and saturation of nuclear forces

2.21. *Mass-defects.* If we except the lightest and the heaviest nuclei, we may say that, disregarding small individual fluctuations, the mass-defects of stable nuclei vary *linearly* with the mass number, i.e. with the total number of constituent nucleons *. The average binding energy per nucleon amounts to about 8 MeV.

The deviation from linearity for heavy nuclei is completely accounted for by the Coulomb repulsion of the protons. Assuming a homogeneous distribution of the charge Ze within a sphere of radius R, the Coulomb energy is given by $\frac{3}{5}\frac{Z^2e^2}{R}$; a somewhat more accurate estimate for small Z would be

$$G = \frac{3}{5}\frac{e^2}{R} Z(Z-1). \tag{1}$$

The Coulomb energy has a marked effect on the relative proportion of neutrons in stable nuclei. There is in fact a general tendency for such nuclei to contain equal numbers of protons and neutrons; but owing to the Coulomb repulsion, the equilibrium, for heavier nuclei, is displaced towards a larger proportion of neutrons. This means that the average binding energy of a given assembly of Z protons and $N = A - Z$ neutrons will depend on the *neutron excess*

$$n = N - Z = A - 2Z \tag{2}$$

in such a way that, for given A, a minimum of the atomic weight (*1.131*) will occur for a certain value of n. The corresponding value Z_A of Z will in general not be an integer; stable nuclei may then exist with the nearest integral numbers of protons (*3.22*).

* See the graphical representation on fig. *2.21* (at the end of the book), based on the numerical data collected by M & F, M & F*.

Besides a "volume" contribution to the proper nuclear energy, proportional to A, a "surface" effect, comparable to the influence of a surface tension, must also be taken into account. This effect will be most important for lighter nuclei and the considerable irregularities presented by very light nuclei may be attributed to its preponderance.

Summing up, the average binding energy per nucleon can be put into the form

$$\varepsilon = -\varepsilon_1 \left[1-\gamma\left(\frac{n}{A}\right)^2\right] + \frac{4\pi R^2 O}{A} + \frac{3}{5}\frac{Z^2 e^2}{AR}, \tag{3}$$

in which the second and third terms express the surface energy (with a surface tension O) and the Coulomb repulsion, respectively. The first term, representing the volume effect, exhibits a parabolic dependence on the relative neutron excess. Inserting in this formula the law (*2.1*–1) of nuclear radii, we get

$$\varepsilon = -\varepsilon_1 \left[1-\gamma\left(\frac{A-2Z}{A}\right)^2\right] + 4\pi r_0^2 O A^{-\frac{1}{3}} + \frac{3}{5}\frac{e^2}{r_0} Z^2 A^{-\frac{1}{3}}. \tag{4}$$

Numerical values of the parameters ε_1, γ, O and r_0 can be found to fit the empirical mass-defect curve (M & F, p. 91—92):

$$\begin{aligned} \varepsilon_1 &= 14{,}66 \text{ MeV}, & \gamma &= 1{,}40, \\ 4\pi r_0^2 O &= 15{,}4 \text{ MeV}, & r_0 &= 1{,}42\cdot 10^{-13} \text{ cm}, \end{aligned} \tag{5}$$

the value of r_0 being in excellent agreement with the independent evidence from nuclear reactions (*2.1*–2). It is useful also to note the value

$$\frac{3}{5}\frac{e^2}{r_0} = 0{,}602 \text{ MeV}. \tag{6}$$

Equation (4) thus gives a semi-empirical representation of remarkable accuracy of the nuclear binding energies.

An idea of the relative importance of the different terms of (4), as well as of the accuracy to be expected, is afforded by the following examples:

Element	Z	A	Volume energy	Surface energy	Coulomb energy	Total binding energy calculated	Total binding energy experimental (M&F*)
Al	13	27	— 14,63	5,13	1,25	— 8,25	— 8,32
Ag	47	109	— 14,26	3,23	2,55	— 8,48	— 8,47
Pb	82	208	— 13,74	2,60	3,28	— 7,86	— 7,83

Energies expressed in MeV

2.22. *Saturation properties of nuclear forces.* The linear variation with mass number of that part of the binding energy due to the proper nuclear forces brings out the fundamental *saturation* property of these forces: it implies in fact that each nucleon is only capable of binding a

fixed number of other nucleons. Considering the short range of the forces, this property throws full light on the law of constant density of nuclear matter (*2.1*). Indeed, as every nucleon interacts only with its nearest neighbours, the structure of a nucleus will be comparable (apart from the scale) with that of ordinary matter in a state (solid or liquid) of high density. The constituent nucleons will on an average be kept apart at a fixed distance, of the order of the range of the force. This is just the law of constant density, which thus yields a value of about $2r_0$ for the *order of magnitude of the range*.

This last inference, as we shall see, is fully confirmed by the detailed analysis of the most direct exploration of the nuclear field which can be carried out, viz. by studying the scattering of protons and neutrons by protons. It is a remarkable fact that the distance $2r_0$, according to (*2.1–2*), nearly coincides with the fundamental length d (*1.21–1*) of electron theory. The significance of this coincidence cannot be understood as long as we have no theory of the ratio of the masses of electron and nucleon. But it is at any rate convenient to make use of it, especially in rapid estimations of orders of magnitude; we shall often avail ourselves of the numerical relation

$$r_0 \approx \tfrac{1}{2} d. \qquad (7)$$

When we now come to ask which kinds of interaction operators are liable to give rise to binding energies exhibiting the saturation property, i.e. increasing — at least in a rough approximation, for a great number of interacting particles — proportionally to this number, we have the choice between two essentially distinct possibilities. In analogy with the case of a liquid, in which the saturation of the atomic binding forces is brought about by the occurrence of a very strong repulsion between two atoms at some critical distance of approach, we might imagine the attractive force between two nucleons at distances of the order of the range to go over, at some smaller distance, into a practically infinite repulsion * (fig. *2.22*).

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* The analogy with a liquid might be deemed imperfect inasmuch as in the case of atoms the complicated distance dependence of the interaction is a consequence of the interplay of simpler forces between the atomic constituents, while in the nuclear case we would have to do with a fundamental law of interaction of complicated form. It must not be overlooked, however, that a relatively simple law of *field* interaction can give rise to a superposition of attractive and repulsive potentials of different ranges· an example of such a meson field theory will be mentioned on a later occasion (*17.1*).

In particular, it has been pointed out by SVARTHOLM [45], Chapter VI, that the many-body interactions occurring in any field theory (*1.34*) might, under suitable conditions, bring about the observed saturation of the nuclear bonds. Although Svartholm's assumptions are admittedly quite arbitrary and over-simplified, his argument may appropriately serve to illustrate the point under discussion. While the two-body attractions will give a nuclear binding proportional to $\binom{A}{2}$, he assumes that the n-body forces will alternatively produce a repulsion or an attraction proportional to $-(-\gamma)^{n-2}\binom{A}{n}$. The

The well-known quantum theory of chemical valency suggests, on the other hand, a mechanism of interaction, the so-called *exchange interaction*, capable of giving saturation effects without involving any explicit assumption of a change of sign of the interaction operator with distance. As a consequence of the exclusion principle, the expectation value of any inter-

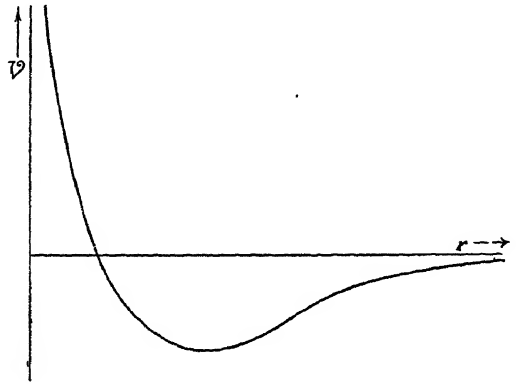

Fig. 2.22. Schematic representation of ordinary potential giving rise to saturation. Interaction energy $\mathcal{V}(r)$ between two nucleons in terms of mutual distance r.

action operator in a stationary state of a system of identical particles, besides "ordinary" terms corresponding to classical effects, involves additional "exchange" terms without classical analogy. It may be seen, quite generally, that for systems of a large number of particles, the exchange terms, in contrast to the ordinary ones, possess the property of saturation.

Let us consider a system of two particles obeying the exclusion principle and suppose that the coupling between them is so weak that it is possible to define individual stationary states $\varphi_n(Q)$ for each particle. The eigenfunctions of the stationary states of the system will then be of the approximate form

$$\psi_{n_1 n_2}(Q^{(1)}, Q^{(2)}) = \frac{1}{\sqrt{2}} \{\varphi_{n_1}(Q^{(1)})\, \varphi_{n_2}(Q^{(2)}) - \varphi_{n_2}(Q^{(1)})\, \varphi_{n_1}(Q^{(2)})\}.$$

The expectation value $\overline{\mathcal{V}}$ of the interaction operator $\mathcal{V}(Q^{(1)}, Q^{(2)})$ in this state becomes, when account is taken of the symmetry of the function $\mathcal{V}$ with respect to $Q^{(1)}$ and $Q^{(2)}$,

$$\overline{\mathcal{V}} = \int \psi^*_{n_1 n_2}\, \mathcal{V}\, \psi_{n_1 n_2} = \int \varphi^*_{n_1}(Q^{(1)})\, \varphi^*_{n_2}(Q^{(2)})\, \mathcal{V}(Q^{(1)}, Q^{(2)})\, \varphi_{n_1}(Q^{(1)})\, \varphi_{n_2}(Q^{(2)}) - \Re \int \varphi^*_{n_1}(Q^{(1)})\, \varphi^*_{n_2}(Q^{(2)})\, \mathcal{V}(Q^{(1)}, Q^{(2)})\, \varphi_{n_2}(Q^{(1)})\, \varphi_{n_1}(Q^{(2)}),$$

---

resulting binding energy per nucleon is

$$\varepsilon = -\varepsilon_1 \cdot \frac{(1-\gamma)^A - 1 + A\gamma}{A\gamma^2};$$

if γ is not too small and A is large, the term $(1-\gamma)^A$ may be neglected, and

$$\varepsilon = -\frac{\varepsilon_1}{\gamma}\left(1 - \frac{1}{A\gamma}\right).$$

This expression exhibits both saturation for large A and increase with A for small A; with $\varepsilon_1 = 4{,}35$ MeV, $\gamma = 0{,}484$, it shows a remarkable (though doubtless fortuitous) agreement with the figures derived from the observed mass-defects.

clearly showing the occurrence of an ordinary and an exchange energy. The saturation properties of the exchange interactions will be extensively discussed in Chapter XI.

For the moment, it will be sufficient to mention the two possibilities of accounting for saturation effects. Later (*17.1*) we shall see how the decision between them is related to the question of the relative importance of non-central couplings in the total nuclear interaction.

2.3. Kinetic effects

The limited range of the nuclear forces implies the occurrence of quite considerable *kinetic energies* of the constituent particles of a nucleus. In fact, under the influence of the short-range interaction with its neighbours, any nucleon will on the average remain confined to a region, the radius of which will be of the order of the range, or somewhat larger; in other words, it will be represented by a wave-packet of linear dimensions $l \gtrsim 2d$. It will then acquire a kinetic energy of the order of magnitude *

$$\varepsilon_{\text{kin}} = \frac{1}{2M}\left(\frac{2\pi\hbar}{l}\right)^2 \tag{1}$$

(this being the smallest eigenvalue of the wave-equation of a particle of mass M with an eigenfunction vanishing on the surface of a sphere of diameter l); for $l = 2d$, we get

$$\varepsilon_{\text{kin}} = \frac{\pi^2}{2}\left(\frac{\hbar}{e^2}\right)^2 \frac{m}{M}\, m \approx 25 \text{ MeV}, \tag{2}$$

and the actual value will perhaps be of the order of 20 MeV.

In spite of this high value of the kinetic energy, the mean velocity of a nucleon, owing to its large mass, is relatively small; in fact, we have, on account of (2),

$$\left(\frac{v}{c}\right)^2 = \frac{2\varepsilon_{\text{kin}}}{M} = \frac{m}{M}\frac{2\varepsilon_{\text{kin}}}{m} \approx \frac{1}{23}. \tag{3}$$

It thus appears that *relativistic effects* should in general not amount to more than a few percent, and that we are justified in basing the discussion of nuclear structure on a non-relativistic treatment of the constituent nucleons. On the other hand, the motions of the nucleons are sufficiently large to suggest that a nucleus should resemble a *liquid droplet* rather than a piece of crystal; as a matter of fact, the liquid phase of Helium at the lowest temperatures (the so-called He II) offers an example of a condensed state of ordinary matter, which is prevented from solidifying (at not too high pressures) by the "zero-point" motion of the atoms.

From another point of view, the kinetic energies of the nucleons are of importance in helping us to picture the interplay of the nuclear energies.

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* Here M denotes the mass of a nucleon, disregarding the small difference $M_n - M_p$.

Comparing the average value of 20 MeV of the kinetic energy with that of the mean binding energy of a nucleon, viz. (discounting the Coulomb repulsion) about 10 MeV, we see that the *potential energy* of the proper nuclear forces must average about 30 MeV per nucleon (a quantity 60 times larger than the Coulomb energy between two protons at distance d). The binding energy is therefore the outcome of a balance between much larger kinetic and potential energies.

This is especially apparent, as pointed out by WIGNER [33a], in the case of the deuteron. In order to keep the kinetic energy lower than the attractive potential, the system, according to (1), has to expand; but, owing to the rapid decrease of the potential for distances larger than the range, this means that the resulting binding energy is very much lowered. It is actually only about 2 MeV, thus really quite small compared to either kinetic or potential energy. As soon, however, as the number of constituent nucleons increases, the situation is considerably altered in favour of the potential energy. Already for the α-particle, the latter energy becomes roughly four or six times * that occurring in the deuteron, while the kinetic energy is only doubled. The particles are then able to settle nearer to each other, letting the nuclear potential, so to speak, display its full strength. The average binding energy per nucleon accordingly rises to a constant value of about 10 MeV.

* According as we assume no forces between nucleons of the same charge, or equal forces between all pairs of nucleons.

CHAPTER III

CHARGE DEPENDENCE OF NUCLEAR FORCES

3.0. After the general survey of the preceding Chapter, we go a step further in the analysis of nuclear properties by looking more closely into the distribution of the stable nuclei according to mass and charge number, and their relation to the neighbouring β-active nuclei. This, combined with the general stability condition *1.131*, will give us important information on the charge dependence of the nuclear forces.

3.1. Proton and neutron pairing

3.11. *Classification of nuclei.* If we classify the stable nuclei according to their mass and charge numbers, we are at once struck by the difference in behaviour between those of even and odd mass number. While the latter are about equally distributed as regards even or odd charge number, the former are divided into two classes of very unequal population: most of them are *even*, i.e. have even numbers both of protons and neutrons; the only *odd* nuclei (i.e. nuclei with odd numbers of protons and neutrons) which are stable are the four lightest of that type, ${}_1^2\mathrm{H}$, ${}_3^6\mathrm{Li}$, ${}_5^{10}\mathrm{B}$, ${}_7^{14}\mathrm{N}$.

A nucleus of mass number A and charge number Z, which therefore contains $N = A - Z$ neutrons, will be denoted either by (A, Z) or by $[N, Z]$. The following table presents the classification just discussed, with the numbers of nuclei of each class, including the α-active ones, according to recent data (M & F):

Class of nuclei	Z	N	Total number *
Even mass nuclei { even nuclei	even	even	177 (15)
Even mass nuclei { odd nuclei	odd	odd	4
Odd mass nuclei	even	odd	60 (8)
	odd	even	52 (1)

* This number includes both stable and α-active nuclei, the number of the latter being indicated between brackets.

3.12. *Proton and neutron pairing.* Apart from the exceptional case of the light odd nuclei, it therefore appears that the successive binding of two nucleons of the same kind (protons or neutrons) sets free a greater amount of energy than that of a proton and a neutron: in other words, there is a tendency for the building up of stable nuclei by successive addition of nucleons to proceed by completion of *pairs* of protons or neutrons. The further empirical fact that all even nuclei have zero angular

momentum receives a simple interpretation by assuming that the two nucleons forming a pair have *opposite spins*. On this view, the "pairing" of like nucleons is thus related to a *"saturation"* of their spins.

If we supposed that individual stationary states could be attributed to the constituent nucleons (*9.21*), we should actually expect a grouping of these states into pairs corresponding to the same specification of all "quantum numbers" (including charge) but to different spin orientations. The structure of a stable nucleus in its ground state would then be vizualized by filling up the successive pairs of individual states with saturated spins by the available protons and neutrons. In general, the "last" pair of neutron states occupied would, on account of the neutron excess, be very different from the last pair of proton states.

3.2. Stability of isobars

3.21. *Stable isobars.* There is another important difference between nuclei of odd and even mass number: while for any odd value of the mass number there is just one stable nucleus, we meet among the even nuclei with a large number of pairs of stable *isobars* (i.e. nuclei of the same mass number). There are in fact 56 pairs and 4 trebles of such isobaric nuclei. In every case, the charge numbers of neighbouring stable isobars differ by two units.

Now, the existence of such isobars cannot be understood solely on the basis of nucleon pairing *. For if two paired nucleons were subject to exactly the same forces (apart from the Coulomb interaction), there could only be one stable nucleus for any even value of the mass number. In fact, if we start from the even nucleus $(2a, 2z)$ and suppose that the stable nucleus of mass $2a + 1$ is obtained by adding a neutron, the next stable nucleus would necessarily be obtained by completing the neutron pair, which gives the nucleus $(2a + 2, 2z)$: the odd isobar $(2a + 2, 2z + 1)$ would have a higher atomic weight, but the even isobar $(2a + 2, 2z + 2)$ a still higher one (fig. *3.21*). In order to reconcile the pairing conception with the empirical evidence, we must therefore, in the first place, assume that two nucleons forming a pair attract each other, so that the atomic weight of the unstable even isobar $(2a + 2, 2z + 2)$ can eventually be sufficiently depressed to reverse the stability relation of this nucleus to the intermediate odd isobar $(2a + 2, 2z + 1)$. This result will only be achieved, however, if it is further assumed that, in the latter isobar, the odd proton and the odd neutron have a much smaller interaction (fig. *3.21*).

The last assumption appears to be generally fulfilled, with the notable exception of the four lightest nuclei of the odd type, which have exactly as many neutrons as protons. In the latter case, the odd proton and the odd neutron are in much the same situation with respect to the other constituent nucleons and may consequently be thought to form a pair entirely comparable to the spin-saturated pairs of nucleons of like charge

* The following argument is taken from B & B, § 10.

discussed above. The roles of spin and charge variables would simply be interchanged. (The angular momenta of the odd nuclei in question * have

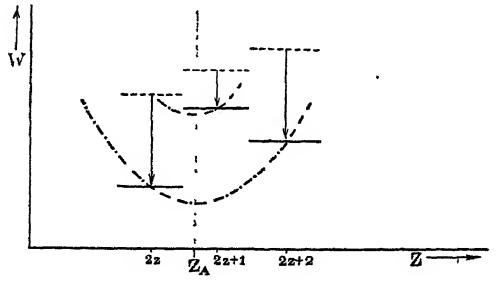

Fig. 3.21. Atomic weight of isobaric nuclei, before (- - -) and after (—) taking account of the interaction between the two last nucleons added in the successive building up of the nuclei.

indeed the value 1, in conformity with the idea that the odd nucleons would have the same spin orientation). We are thus led to assume that such pairs of nucleons also exert a strong attraction on each other, so that the odd isobar then becomes more stable than the neighbouring even nuclei.

Using the more definite picture of *3.12*, based on the conception of individual states of the constituent nucleons, we may express the difference between the light and the heavier odd nuclei by stating that the odd nucleons experience a strong mutual attraction when they belong to states of identical specification except for the charge variable, whereas they do not interact appreciably when, owing to the neutron excess, they occupy different states.

On the whole, we thus arrive, at least qualitatively, at a complete account of the main features of the distribution of stable nuclei; and the analysis has disclosed that *besides the proton-neutron forces, there are also forces between like particles, which are attractive for "paired" nucleons.*

3.22. *Atomic weights of isobars.* To get some idea of the order of magnitude of the energies involved in the preceding considerations on isobar stability, we first observe that our general formula (*2.21*–4) must be made more precise in order to take account of the differences between odd and even mass and charge numbers. Consider an *even* nucleus (A, Z); according to formula (*2.21*–4), its atomic weight may be written in the form

$$W(A, Z) = W_A + w_A (Z - Z_A)^2, \tag{1}$$

with

$$Z_A = \frac{4\varepsilon_1 \gamma + (M_n - M_p - m)}{2 w_A}, \qquad w_A = \frac{4\varepsilon_1 \gamma}{A} + \frac{3}{5}\frac{e^2}{r_0} A^{-\frac{1}{3}}; \tag{2}$$

the points representing stable isobars of mass number A and *even* Z in a graph like fig. *3.21* with coordinates Z, W lie on a parabola defined by

* The angular momentum of $^{10}_{5}B$ is not known.

equation (1). The *odd* isobars of the same mass number, however, correspond to another parabola, given by

$$W(A, Z+1) = W_A + \delta_A + w_A (Z+1-Z_A)^2 \tag{3}$$

with a suitably chosen additive constant δ_A. The difference

$$\zeta = Z+1-Z_A \tag{4}$$

between the charge number of the odd isobar and the abcissa Z_A of the apices of the parabolas is obviously comprised between $+1$ and -1; with this notation, we have

$$\begin{aligned} W(A, Z) &= W_A + w_A (\zeta-1)^2 \\ W(A, Z+1) &= W_A + \delta_A + w_A \zeta^2 \\ W(A, Z+2) &= W_A + w_A (\zeta+1)^2. \end{aligned} \tag{5}$$

Now, if there were no like-particle forces, $W(A, Z+1)$ would be just midway between $W(A, Z)$ and $W(A, Z+2)$, i.e. δ_A would be equal to w_A. The effect of the like-particle forces is therefore expressed by $\delta_A - w_A$. The conditions for the odd isobar to be unstable with respect to both even isobars can be put into the form

$$\delta_A - w_A > 2 w_A |\zeta|. \tag{6}$$

This inequality defines a critical value

$$\zeta_{\text{crit}} = \frac{\delta_A - w_A}{2 w_A}, \tag{7}$$

such that there is more than one stable isobar of mass number A if and only if

$$|\zeta| < \zeta_{\text{crit}}. \tag{8}$$

If we assume that the values of ζ are distributed at random along the whole range of even mass numbers, the statistics of isobaric pairs (or trebles) will immediately give us an estimate of some average value $\bar{\zeta}_{\text{crit}}$. Since we encounter 60 isobaric pairs or trebles among the 97 even mass numbers between 16 and 208, we may take

$$\bar{\zeta}_{\text{crit}} \approx 0{,}6. \tag{9}$$

With the numerical values (*2.21–5, 6*), giving

$$w_A = \left(\frac{82{,}1}{A} + \frac{0{,}6}{A^{\frac{1}{3}}} \right) \text{MeV}, \tag{10}$$

we accordingly find

$$\delta_A - w_A \approx 2 w_A \bar{\zeta}_{\text{crit}} \approx 1{,}2\, w_A. \tag{11}$$

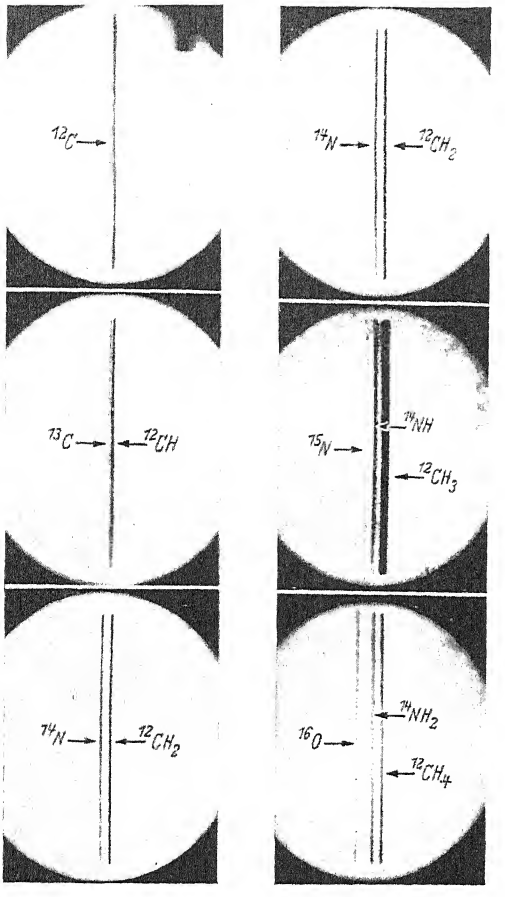

Fig. 3.221–3. Mass-spectrograms showing the successive addition of two neutron-proton pairs to the ^{12}C nucleus (MATTAUCH [37]).

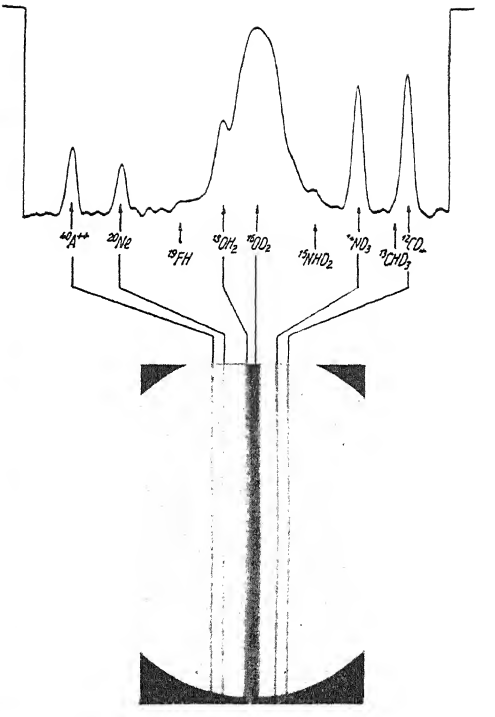

Fig. 3.221–4. Mass-spectrogram, pertaining to mass-number 20, showing binding energies of various aggregates of protons and neutrons (MATTAUCH [39]).

This gives, e.g.:

A	w_A MeV	$\delta_A - w_A$ MeV	δ_A MeV
64	1,4	1,7	3,1
125	0,8	0,9	1,7
216	0,5	0,6	1,1

3.221. *Comparison with measurements of mass-defects.* Our knowledge of accurate atomic weights is still so fragmentary that there is as yet only one instance of isobaric nuclei for which the relations between atomic weights embodied in fig. *3.21* can be checked with reasonable accuracy. It concerns the mass-number $A = 64$, for which we have (M & F*, table I):

Z		$W - A$ 10^{-4} MU
28	Ni (stable)	$-525{,}6 \pm 5{,}6$
29	Cu (unstable)*	$-507{,}4 \pm 5{,}6$
30	Zn (stable)	$-513{,}6 \pm 5{,}6$

(The *relative* accuracy of these three figures is better than the indicated absolute uncertainty).

As to the atomic weights of lighter nuclei (up to ${}^{40}_{18}\mathrm{A}$), which have been determined with high accuracy for almost all unstable as well as stable isotopes, they exhibit such large individual fluctuations that it is not possible at this stage to draw from them other than qualitative conclusions. The mass-defects of the stable nuclei strikingly bring out the successive completion of neutron and proton pairs, with the resulting periodic occurrence of nuclei of maximum binding energy, composed of equal even numbers of protons and neutrons.

We may further consider sequences of (stable and unstable) *isotopes* (i.e. nuclei with the same number of protons) or *isotones* (i.e. nuclei with the same number of neutrons) and derive from the comparison of their mass-defects the binding energy of the "last" nucleon serving to build up any one of these nuclei from the preceding one of the sequence. The result is given (at the end of the book) in fig. *3.221–1*, taken from M & F*, and further illustrated by a diagram (fig. *3.221–2*). Both directly show how in all cases the second nucleon of a given pair is more tightly bound than the first **.

3.3. Charge symmetry of nuclear forces

A further feature of nuclear interaction which can be established with fair accuracy is the following: *the proper nuclear forces are symmetrical with respect to the charges of the interacting nucleons.* This property can

* ^{64}Cu transmutes as well into ^{64}Ni (with positon emission) as into ^{64}Zn (with negaton emission).

** Plate II reproduces some beautiful mass-spectrograms, obtained by MATTAUCH [37, 39], to the same effect. A more detailed discussion of this and other general evidence is given by JENSEN [39].

be inferred, in the first place, from the tendency of stable nuclei toward equality of proton and neutron numbers, especially apparent in the light nuclei from ${}_{2}^{4}\mathrm{He}$ to ${}_{8}^{16}\mathrm{O}$, where it is not yet disturbed by the influence of the Coulomb force. The nuclei in question are formed by alternate addition of a neutron and a proton, which corresponds to an alternation of the two types of nucleon pairing discussed above (*3.21*). The only effect of the Coulomb repulsion is here to give the binding of a neutron to an even nucleus a slight preference over that of a proton.

Still more definite evidence for the symmetry property under consideration is provided by the light nuclei with a neutron excess -1, which transmute with positon emission into isobaric nuclei of neutron excess $+1$; there is a long sequence of known instances of this type of β-transitions, beginning with ${}_{6}^{11}\mathrm{C} \rightarrow {}_{5}^{11}\mathrm{B}$, ${}_{7}^{13}\mathrm{N} \rightarrow {}_{6}^{13}\mathrm{C}$, etc. and extending up to ${}_{21}^{41}\mathrm{Sc} \rightarrow {}_{20}^{41}\mathrm{Ca}$. In fact (FOWLER, DELSASSO and LAURITSEN [36]), if we assume the symmetry of the nuclear forces with respect to the charges of the nucleons, the binding energies of isobaric nuclei such that the number of protons of the one is the number of neutrons of the other will differ only by the electrostatic contributions, which may easily be estimated; and in the special case of the above mentioned isobaric pairs with neutron excesses ± 1, this energy difference can directly be deduced from the maximum kinetic energy K_β of the emitted positons. If we take for the Coulomb energy the expression (*2.22–1*), equation (*1.13–1*) for the mass difference of the isobaric pair takes the form

$$\frac{3}{5}\frac{e^2}{R}(A-1) = K_\beta + (M_n - M_p + m). \tag{1}$$

We can use this equation either in connexion with the expression (*2.1–1, 2*) for the nuclear radius to predict the limiting energy K_β of the positon spectrum, or conversely to calculate the nuclear radius with the help of the observed K_β values. In any case, the remarkable consistency of the results * affords strong support to the underlying assumption about the symmetry of the nuclear forces.

The same argument also applies to the pairs **

$${}_{2}^{3}\mathrm{He} \xleftarrow{\beta^-} {}_{1}^{3}\mathrm{H}, \qquad {}_{4}^{7}\mathrm{Be} \xrightarrow{K} {}_{3}^{7}\mathrm{Li}, \qquad {}_{5}^{9}\mathrm{B} - {}_{4}^{9}\mathrm{Be},$$

in which the unstable partner is not a positon emitter; the mass differences can in these cases be determined by other methods. However, a more precise calculation of the Coulomb energy is required in these cases to account for fluctuations in the observed mass differences with the parity of Z: these may be due, partly to a contribution from the exchange term

* The available evidence, including the comparison with formula (1) is collected in table *A 1.21*. Systematic investigations on this point are due, above all, to WHITE *et al.* [39, 41]; see also ELLIOT and KING [41].

** The nucleus ${}_{5}^{9}\mathrm{B}$ disintegrates into two α-particles and a proton.

(*2.22*) of Coulomb energy, which, according to the parity of Z, enhances or counteracts the ordinary term (PHILLIPS and FEENBERG [41]) *, partly to variations of nuclear dimensions according to the more or less loose binding of the "last" nucleon (BETHE [38a], ELLIOT and KING [41]). When account is taken of these corrections, a further confirmation of the above conclusion is obtained. It is interesting to note that, apart from the mentioned fluctuations, the nuclear radius is assumed to satisfy the law (*2.1*–1) of constant nuclear density, implying that there is no "blowing up" of the nucleus by Coulomb repulsion.

We must finally mention the pair of unstable isobars ^{10}C–^{10}Be (which both go over by β^-- or β^+-decay into the stable odd nucleus ^{10}B) as the one known example of a pair of isobars with neutron excesses -2 and $+2$, for which it could be verified that the energy difference is entirely accounted for by electrostatic interactions (WHITE *et al.* [40]). In this case, the mass of ^{10}C is deduced from that of ^{10}B and the maximum energy of the positon spectrum, while that of ^{10}Be is brought back to that of ^{9}Be by the study of the reaction ^{9}Be (d, p) ^{10}Be.

Summing up, we are led to a precise formulation of the symmetry of proper nuclear forces with respect to charge: *the neutron-neutron interaction operator is the same as that for two protons (exclusive of the Coulomb force)*. This conclusion is important inasmuch as there is as yet no more direct approach to the force between two neutrons (*14.0*).

3.4. Building up of stable nuclei

3.40. It is a remarkable fact, especially stressed by FUCHS [39], that the mainly qualitative considerations developed in this first Part are already sufficient to convey a general understanding of the apparently capricious order in which stable nuclei of the different classes — even nuclei, odd nuclei, odd mass nuclei — successively appear when the numbers of constituent protons and neutrons are increased. In this section, we shall analyse the distribution of the known stable nuclei from the point of view put forward in Fuchs' paper.

3.41. *The energy surface.* As already pointed out in the case of even and odd nuclei (*3.22*), the atomic weights, and therefore also the binding energies $\mathcal{E}(N, Z)$ (< 0), of the nuclei of the three classes are quite distinct functions of the numbers N, Z of neutrons and protons. In a graphical representation of the binding energy in terms of N and Z, the points pertaining to nuclei of these various classes accordingly lie on three distinct surfaces, which may be regarded as three sheets of the *energy surface* $\mathcal{E}(N, Z)$. The sheet corresponding to odd nuclei lies above that for even nuclei, except in the region of the lightest nuclei, where a crossing over of the two sheets occurs. The sheet pertaining to odd mass nuclei lies somewhere in between the other two.

* A more general discussion of this effect will be given in *10.34*.

We shall restrict the following discussion to the region in which the neutron excess of the stable nuclei is larger than 2, thus excluding the region of the lightest nuclei, for the treatment of which we refer the reader to Fuchs' paper. In the region of heavier nuclei, all three sheets of the energy surface (see fig. *3.41* at the end of the book) have similar properties: each one is moulded into a "valley", sloping down from the high mass number values, and in or near the bottom of which lie the stable nuclei. The projection of the bottom of the valley on the (N, Z)-plane — the *line of maximum binding* — is (as first noted by GAMOW [34]) a rather winding line. An explanation of such irregularities would necessitate a much closer investigation of nuclear structure than is at present possible. Here, however, we shall only be concerned with the general orientation of this line, defined by an angular coefficient

$$\frac{dN}{dZ} \approx 1{,}5 \tag{1}$$

(dN, dZ denoting the increments of N and Z along the line).

Cross-sections of the energy surface by planes $N + Z = \text{const}$, i.e. loci of isobaric nuclei, are curves turning their concavity upward; their lowest parts are approximated by parabolas similar to those of *3.22* (which referred to atomic weights instead of binding energies). At the lowest point, projecting on the line of maximum binding, one has

$$\frac{\partial \mathcal{E}}{\partial Z} = \frac{\partial \mathcal{E}}{\partial N}, \qquad \frac{\partial^2 \mathcal{E}}{\partial N^2} - 2\frac{\partial^2 \mathcal{E}}{\partial Z\, \partial N} + \frac{\partial^2 \mathcal{E}}{\partial Z^2} > 0. \tag{2}$$

Now, the saturation character of the nuclear bonds conditions further general features of the sheets of the energy surface. First, the slope of the valley will be nearly uniform along the whole line of maximum binding (in the region of heavy nuclei we are considering), so that

$$\frac{\partial^2 \mathcal{E}}{\partial N^2}\, dN + \frac{\partial^2 \mathcal{E}}{\partial N\, \partial Z}\, dZ = 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial Z\, \partial N}\, dN + \frac{\partial^2 \mathcal{E}}{\partial Z^2}\, dZ = 0. \tag{3}$$

Combined with the inequality (2), this shows, in particular, that

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} > 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial Z^2} > 0 \quad , \quad \frac{\partial^2 \mathcal{E}}{\partial N\, \partial Z} < 0 \tag{4}$$

along the line of maximum binding. Secondly, since the relative increase of the binding of a neutron (say) when another nucleon is added to the nucleus is very small,

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} \ll \left|\frac{\partial \mathcal{E}}{\partial N}\right|, \qquad \left|\frac{\partial^2 \mathcal{E}}{\partial N\, \partial Z}\right| \ll \left|\frac{\partial \mathcal{E}}{\partial N}\right|. \tag{5}$$

As a matter of fact, the rough estimate allowed by the experimental data yields

$$\frac{\partial^2 \mathcal{E}}{\partial N^2} \approx 0{,}5 \ldots 1 \text{ MeV}, \tag{6}$$

while according to (3), (1),

$$\left|\frac{\partial^2 \mathcal{E}}{\partial Z\,\partial N}\right| \approx 1{,}5\,\frac{\partial^2 \mathcal{E}}{\partial N^2}; \tag{7}$$

these values are to be compared with $|\partial \mathcal{E}/\partial N| \approx 8$ MeV.

3.42. *Stability conditions.* We now go over, following Fuchs, to a closer discussion of the stability condition *1.131*. Fuchs' argument is essentially based on the consideration of the amounts of energy released when a specified nucleon is added to the nucleus. We have to distinguish four such energy quantities, according as the added nucleon is a neutron or a proton and according as it is the first or the second of the pair which is being formed; for an initial nucleus $[N, Z]$ containing N neutrons and Z protons, we use the self-explanatory notations $\partial_N^{(1)}(N, Z)$, $\partial_N^{(2)}(N, Z)$, $\partial_Z^{(1)}(N, Z)$, $\partial_Z^{(2)}(N, Z)$, with the convention that the ∂_Z's do *not* include the change in Coulomb energy. Just as we did with $\mathcal{E}(N, Z)$, we regard the ∂'s as continuous functions defined for all real values of their arguments. We list here some general properties of the ∂'s:

(1) On account of (5), ∂_N is not very different from $-\partial \mathcal{E}/\partial N$ and consequently, by (4), decreases with increasing N when Z remains unchanged. Similarly, ∂_Z is found to vary in the opposite direction; summing up:

$$\text{for fixed } Z, \begin{matrix}\partial_N(N,Z) \text{ decreases} \\ \partial_Z(N,Z) \text{ increases}\end{matrix} \text{ with increasing } N \text{ (or } n). \tag{8}$$

(2) The attraction between paired nucleons implies

$$\partial_N^{(2)} > \partial_N^{(1)} \quad , \quad \partial_Z^{(2)} > \partial_Z^{(1)}. \tag{9}$$

(3) Considering that the attraction just mentioned will mainly depend on the distance of the two interacting nucleons and that this distance will on the average remain constant whatever the total number of constituent nucleons, the differences $\partial_N^{(2)} - \partial_N^{(1)}$, $\partial_Z^{(2)} - \partial_Z^{(1)}$ will have the same property:

$$\partial_N^{(2)} - \partial_N^{(1)},\ \partial_Z^{(2)} - \partial_Z^{(1)} \text{ are practically independent of } N, Z. \tag{10}$$

(4) On account of the symmetry of nuclear forces with respect to charge, one has moreover

$$\partial_N^{(2)} - \partial_N^{(1)} \approx \partial_Z^{(2)} - \partial_Z^{(1)}. \tag{11}$$

For the order of magnitude, we may roughly take

$$\partial_N^{(2)} - \partial_N^{(1)} \approx 5\ \text{MeV}. \tag{12}$$

With the help of the ∂'s, the stability condition *1.131* is readily brought into the following form: the former or the latter of the nuclei $[N, Z]$, $[N-1, Z+1]$ is stable against β-transitions according as the difference

$\partial_Z(N-1, Z) - \partial_N(N-1, Z)$ is smaller or larger than a critical value defined by the equation

$$\partial_Z(N-1, Z) - \partial_N(N-1, Z) = \frac{6}{5}\frac{e^2}{r_0}\frac{Z}{(N+Z)^{\frac{1}{3}}} - (M_n - M_p - m). \quad (13)$$

If we regard the charge number Z as given, this equation defines a corresponding critical value (generally not integral) of N, or of the neutron excess n. Choosing now *even* values for N and Z, and grouping together the nuclei of charge numbers Z, $Z+1$ and mass numbers $N+Z-1$, $N+Z$, we thus get for each group * four critical values of the neutron excess, viz.

n_1		$[N-1, Z+1]$	,	$[N-2, Z+2]$
n_2	separating the domains	$[N-1, Z]$	,	$[N-2, Z+1]$
n_3	of stability of the nuclei	$[N-2, Z+1]$	,	$[N-3, Z+2]$
n_4		$[N, Z]$	,	$[N-1, Z+1]$.

Because of (8), the nuclei in the left / right column are stable when the actual neutron excess is smaller / larger than the corresponding critical n_i, and we shall presently see that

$$n_1 < n_2 < n_3 < n_4. \quad (14)$$

With increasing neutron excess, we therefore successively meet, for the charge numbers Z, $Z+1$, $Z+2$, the following types of stable nuclei:

n ↓	$[N-1, Z]$	—	—
	$[N, Z]$	—	—
n_1			
	$[N-1, Z]$	—	—
	$[N, Z]$	—	$[N-2, Z+2]$
n_2			
	—	$[N-2, Z+1]$	—
	$[N, Z]$	—	$[N-2, Z+2]$
n_3			
	—	—	$[N-3, Z+2]$
	$[N, Z]$	—	$[N-2, Z+2]$
n_4			
	—	—	$[N-3, Z+2]$
	—	—	$[N-2, Z+2]$

(Vertical columns contain isotopes, horizontal rows isobars.) A representation of all known stable (and α-active) nuclei, according to this scheme, will be found at the end of the book (fig. *3.42*).

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* Equation (13) gives rise to positive critical values of n only when the Coulomb energy variation on the right becomes sufficiently large, i.e. when Z exceeds a certain limit. For this point, and the treatment of lighter nuclei, see Fuchs' paper. Our discussion remains restricted to the heavier nuclei.

3.43. *Properties of the critical values of the neutron excess.* With a view to establishing the inequalities (14), we observe that the critical n-value resulting from an equation of the type (13) is larger, the smaller the left-hand side; on the other hand, the right-hand side is quite insensible to changes of Z or $N+Z$ by one or two units: it will therefore suffice to order the left-hand side quantities

$$\begin{aligned} \delta_1 &= \{\delta_Z^{(2)} - \delta_N^{(1)}\}_{N-2,Z+1} \quad , \quad \delta_2 = \{\delta_Z^{(1)} - \delta_N^{(1)}\}_{N-2,Z} \\ \delta_3 &= \{\delta_Z^{(2)} - \delta_N^{(2)}\}_{N-3,Z+1} \quad , \quad \delta_4 = \{\delta_Z^{(1)} - \delta_N^{(2)}\}_{N-1,Z}, \end{aligned} \tag{15}$$

corresponding to the various n_i's. Comparing (11), (12) with (6), (7), we conclude that the sign of any difference $\delta_i - \delta_j$ involving a difference $\delta_Z^{(2)} - \delta_Z^{(1)}$ or $\delta_N^{(2)} - \delta_N^{(1)}$ will be fixed by the sign of this last difference, according to (9), irrespective of other small differences of $\delta^{(1)}$'s or $\delta^{(2)}$'s with different arguments. This yields immediately

$$\delta_1 > \delta_2 > \delta_4, \qquad \delta_1 > \delta_3 > \delta_4. \tag{16}$$

In order to compare δ_2 and δ_3, we first observe that

$$\delta_2 \approx \delta_3 (N+2), \tag{17}$$

where the notation $\delta_3(N+2)$ is used to represent the value $\{\delta_Z^{(2)} - \delta_N^{(2)}\}_{N-1,Z+1}$ of the function $\delta_3(N)$ for the value $N+2$ of its argument. Relation (17) follows from (10), (11) with the help of the identity

$$\delta_Z^{(1)}(N-2,Z) + \delta_N^{(1)}(N-2,Z+1) \equiv \delta_N^{(1)}(N-2,Z) + \delta_Z^{(1)}(N-1,Z), \tag{18}$$

expressing that the energy change due to the addition of a proton and a neutron to the nucleus is independent of the order in which the particles are added. In virtue of (8), one gets from (17) the inequality

$$\delta_2 > \delta_3, \tag{19}$$

which, together with (16), leads to the inequalities (14).

Moreover, using (10) and (11), one readily finds

$$\begin{aligned} &\delta_1 - \delta_2 \approx \delta_3 - \delta_4 \\ &\delta_1 - \delta_4 \approx 2\left[\delta_N^{(2)} - \delta_N^{(1)} - \left|\frac{\partial^2 \mathcal{E}}{\partial Z\, \partial N}\right|\right]. \end{aligned} \tag{20}$$

From (17) and (20) estimates of the differences $n_i - n_j$ can readily be obtained, if one again disregards possible differences between the Coulomb energy changes occurring on the right-hand side of (13). One finds

$$\begin{aligned} &n_2 - n_1 \approx n_4 - n_3 \\ &n_3 - n_2 \approx 2 \\ &n_4 - n_1 \approx 2\,\frac{\delta_N^{(2)} - \delta_N^{(1)} - \left|\dfrac{\partial^2 \mathcal{E}}{\partial Z\, \partial N}\right|}{\dfrac{\partial^2 \mathcal{E}}{\partial N^2} + \left|\dfrac{\partial^2 \mathcal{E}}{\partial Z\, \partial N}\right|}; \end{aligned} \tag{21}$$

with the help of (7), (6), the last approximate equality yields

$$n_4 - n_1 \approx 1{,}2 \left(\frac{\partial_N^{(2)} - \partial_N^{(1)}}{\left| \frac{\partial^2 \mathcal{E}}{\partial Z \, \partial N} \right|} - 1 \right) \tag{22}$$

$$n_4 - n_1 \approx 3 \ldots 7; \tag{23}$$

then, from the first two relations (21) and (23),

$$n_2 - n_1 \approx n_4 - n_3 \approx 0{,}5 \ldots 2{,}5. \tag{24}$$

The difference $n_3 - n_2$ characterizes the "isotopic breadth" (number of stable isotopes of given Z) of nuclei of odd charge; $n_4 - n_1$ indicates, for any given Z, how many pairs of stable isobars of charge numbers Z, $Z + 2$ can exist.

The relations (21), (23), (24) are on the whole in good agreement with the empirical n_i's, which can be determined from the table of known nuclei, for the different (even) values of Z, with a margin of 2 (or in some cases only 1) *. In particular, the number of stable isotopes of any odd charge number is always 1 or 2, corresponding to $2 \leqq n_3 - n_2 \leqq 4$. The difference $n_4 - n_1$ is found to increase with Z up to $Z \approx 60$, and then to decrease; the first trend may be understood from (22), while the explanation of the decrease at higher Z is not clear (cf. Fuchs' paper).

* The ∂_i's can then be deduced in terms of Z from the equations (13). Owing to the slowness of the variation of the right-hand sides of these equations with n, all four ∂_i's come to lie practically on the same smooth curve, slowly rising with increasing Z. The orders of magnitude range from $\approx$ 3 MeV ($Z \approx 10$) to $\approx$ 11 MeV ($Z \approx 80$).

PART II

TWO-NUCLEON SYSTEMS ON THE HYPOTHESIS OF CENTRAL INTERACTION

CHAPTER IV

DYNAMICAL VARIABLES AND FUNDAMENTAL EQUATIONS

4.1. Dichotomic variables

4.11. *Definition and main properties.* When we have to distinguish between two possible states of some system, such as, e.g., the two states of different charge (proton and neutron) of a nucleon, we may conveniently characterize them by the eigenvalues, say $+1$ and -1, of some quantal variable τ. A discrimination between the two states in question then corresponds to a "measurement" of the variable τ, such a measurement yielding the answer $+1$ or -1. In a matrix representation, such a *dichotomic variable* will appear as a Hermitian matrix with 2 rows and 2 columns, and will satisfy the equation

$$\tau^2 = 1. \tag{1}$$

From these conditions, it is easily deduced that the most general form for τ is *

$$\tau = \begin{pmatrix} a & \sqrt{1-a^2}\cdot e^{-i\varphi} \\ \sqrt{1-a^2}\cdot e^{i\varphi} & -a \end{pmatrix}, \tag{2}$$

where a is a real number with $|a| \leq 1$, and $e^{\pm i\varphi}$ represents an arbitrary phase factor. Putting $a = \cos\vartheta$, we may also write

$$\tau = \begin{pmatrix} \cos\vartheta & \sin\vartheta\cdot e^{-i\varphi} \\ \sin\vartheta\cdot e^{i\varphi} & -\cos\vartheta \end{pmatrix}. \tag{3}$$

This form discloses a remarkable connexion between our dichotomic variables and the directions in ordinary space. In fact, we may interpret (ϑ, φ) as the polar coordinates (with respect to an arbitrarily chosen system of reference) of a unit vector $\vec{n}$, the cartesian components of which will thus be

$$\begin{aligned} n_x &= \sin\vartheta\cos\varphi \\ n_y &= \sin\vartheta\sin\varphi \\ n_z &= \cos\vartheta. \end{aligned} \tag{4}$$

Instead of (3) we may then write

$$\tau = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}. \tag{5}$$

* We exclude, of course, the unit matrix, which is also a solution of (1).

Introducing now the matrices

$$\tau_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

we see from (5) that the special dichotomic variables which correspond to the directions of the x-, y- and z-axis, respectively, and which we will denote by τ_x, τ_y, τ_z, are represented just by these matrices:

$$\tau_x = \tau_1, \quad \tau_y = \tau_2, \quad \tau_z = \tau_3; \tag{7}$$

further the relation

$$\tau = \tau_x n_x + \tau_y n_y + \tau_z n_z \tag{8}$$

holds between the variable τ and the unit vector $\vec{n}$.

If we choose another system of reference to represent the unit vector $\vec{n}$, the matrices τ_x, τ_y, τ_z will no longer be given by (7); in fact, as is immediately apparent from the relation (8), they will transform *like the components of a vector*. The variable τ, as expressed by (8), may thus be regarded as the projection of the fundamental vector $\vec{\tau}$ with components τ_x, τ_y, τ_z on the direction $\vec{n}$; as such, it will be denoted as $\tau(\vec{n})$. For the product $\tau(\vec{n})\,\tau(\vec{n'})$ we derive from (5), by a straightforward calculation, the invariant relation

$$\tau(\vec{n})\,\tau(\vec{n'}) = \vec{n}\cdot\vec{n'} + i\tau(\vec{n}\wedge\vec{n'}), \tag{9}$$

from which it follows that

$$[\tau(\vec{n}),\ \tau(\vec{n'})]_+ = 2\vec{n}\cdot\vec{n'} \tag{10}$$

$$[\tau(\vec{n}),\ \tau(\vec{n'})]_- = 2i\tau(\vec{n}\wedge\vec{n'}). \tag{11}$$

Applied to the cartesian (orthogonal) components of the vector $\vec{\tau}$, these relations become

$$\tau_x\tau_y = i\tau_z \quad \text{(cycl.)} \tag{12}$$

$$[\tau_x, \tau_y]_+ = 0 \quad \text{(cycl.)} \tag{13}$$

$$[\tau_x, \tau_y]_- = 2i\tau_z \quad \text{(cycl.)}. \tag{14}$$

On account of the special representation (6), analogous formulae may be written down for the matrices τ_1, τ_2, τ_3; they may of course, be directly verified with the help of the definitions (6).

4.111. *Spin.* The correspondence here discussed between dichotomic variables and spatial directions is, in the general case, purely symbolical. But it acquires a real significance in the theory of the *spin*, or intrinsic

angular momentum, of the electron or nucleon. For the establishment of this concept the existence of a set of dichotomic variables forming a vector, together with the relations (14) between them, is of essential importance. As is well-known, spectroscopic evidence showed the necessity of attributing to the electron an intrinsic angular momentum, such that its component in any direction can only have two values of equal magnitudes and opposite signs. This spin component must thus clearly be proportional to a dichotomic variable like $\tau(\vec{n})$, which we shall call $\sigma(\vec{n})$. The coefficient expressing its magnitude can further be determined by the condition that the commutation rules

$$[S_x, S_y]_- = i\hbar S_z \quad \text{(cycl.)}, \tag{15}$$

characteristic for the components of any angular momentum, be satisfied. Comparing (15) with (14), we see that this is just the case for the components of any dichotomic vector $\vec{\tau}$, apart from the factor $\frac{1}{2}\hbar$. The spin angular momentum $\vec{S}$ may thus be expressed as

$$\vec{S} = \tfrac{1}{2}\hbar\vec{\sigma}. \tag{16}$$

4.12. *Transformation of eigenfunctions.* When the description of a system involves a dichotomic variable τ, the wave-functions $\psi(\tau')$ depend on the eigenvalue τ' of τ; since τ' is susceptible of the two values $+1$ and -1, we may conceive the wave-function as consisting of two "components"

$$u \equiv \psi(+1) \quad , \quad v \equiv \psi(-1),$$

which, in connexion with the matrix representation of the dichotomic variables, are conveniently arranged as a matrix

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{17}$$

with two rows and one column. When operators involving dichotomic variables are written as matrices, the result of their operating on ψ is then obtained by applying the usual rule for matrix multiplication. The quantities $|u|^2$, $|v|^2$ give the relative probabilities for finding the state corresponding to $\tau' = +1$ or $\tau' = -1$, respectively (all other conditions being the same). The eigenfunctions of these states are therefore of the form

$$\psi_+ = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad , \quad \psi_- = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{18}$$

respectively; the matrix τ itself has the diagonal form denoted, according to (6), by τ_3. The functions u and v depend, of course, on the other variables of the system; in particular, there are many different eigenfunctions of the type ψ_+ or ψ_-.

Since τ has the form τ_3, the operator which, applied to any wave-function ψ, transforms it into an eigenfunction ψ_+ is

$$\mathsf{T}_+ = \tfrac{1}{2}(1 + \tau_3); \tag{19}$$

similarly,

$$\mathsf{T}_- = \tfrac{1}{2}(1 - \tau_3) \tag{20}$$

transforms ψ into an eigenfunction ψ_-. Especially, we have, by (18),

$$\mathsf{T}_+ \psi_- = 0 \quad , \quad \mathsf{T}_- \psi_+ = 0, \tag{21}$$

and conversely, these relations characterize the eigenfunctions of the states $\tau' = +1$ and $\tau' = -1$, respectively. Let now $\tau_\perp$ be the dichotomic variable corresponding to an arbitrary direction perpendicular to that relating to the variable τ; in our representation, in which $\tau = \tau_3$, this will be, according to (7),

$$\tau_\perp = \tau_1 \cos\varphi + \tau_2 \sin\varphi, \tag{22}$$

with arbitrary φ. Since, by (10), $\tau_\perp$ anticommutes with τ, we have

$$\tau_\perp \mathsf{T}_+ = \mathsf{T}_- \tau_\perp. \tag{23}$$

Observing that $\mathsf{T}_+\mathsf{T}_- = 0$, we get from this

$$\mathsf{T}_+ \tau_\perp \mathsf{T}_+ = 0,$$

which means, according to (21), that if we apply the operator $\tau_\perp \mathsf{T}_+$ to any eigenfunction ψ_+, the result will be an eigenfunction ψ_- (while the application of $\tau_\perp \mathsf{T}_+$ to an eigenfunction ψ_- gives zero). In other words, the operator $\tau_\perp \mathsf{T}_+$ effects the transition from the state $\tau' = +1$ to the state $\tau' = -1$. Denoting it by Π, we have by (22)

$$\Pi = \tau_\perp \mathsf{T}_+ = \tfrac{1}{2}(\tau_1 - i\tau_2)\, e^{i\varphi},$$

or more simply, if we choose $\varphi = 0$,

$$\Pi = \tfrac{1}{2}(\tau_1 - i\tau_2). \tag{24}$$

Similarly, the operator transforming an eigenfunction ψ_- into one of the type ψ_+ is $\tau_\perp \mathsf{T}_-$ or $\Pi^\dagger$.

4.13. *Exchange operators for two identical particles.* Let us now consider two identical systems (1), (2), such as, for instance, two electrons or two nucleons; if a dichotomic variable τ, and consequently the dichotomic vector $\vec{\tau}$, is attached to the system, let $\vec{\tau}^{(1)}$ be the vector pertaining to system (1) and $\vec{\tau}^{(2)}$ the analogous vector pertaining to system (2). Operators involving both $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$ may be written as matrices with 4 rows

and 4 columns, the elements of which are numbered by the eigenvalues ± 1 of $\tau_3^{(1)}$ and $\tau_3^{(2)}$; thus, for instance,

$$\tau_1^{(1)}\tau_1^{(2)} = \begin{pmatrix} 0 & \tau_1^{(2)} \\ \tau_1^{(2)} & 0 \end{pmatrix} = \begin{array}{cc} \tau_3^{(1)\prime} & \\ & \tau_3^{(2)\prime} \\ +1 & +1 \\ +1 & -1 \\ -1 & +1 \\ -1 & -1 \end{array} \begin{array}{cccc} +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}. \tag{25}$$

The wave-functions of the total system (1) + (2) will then consist of 4 "components", numbered in the same way and forming a 1-column matrix.

Especially interesting are the operators $\vec{\tau}^{(1)}\vec{\tau}^{(2)}$ and

$$P = \tfrac{1}{2}(1 + \vec{\tau}^{(1)}\vec{\tau}^{(2)}). \tag{26}$$

Using (1) and (12), we readily find

$$(\vec{\tau}^{(1)}\vec{\tau}^{(2)})^2 = 3 - 2(\vec{\tau}^{(1)}\vec{\tau}^{(2)}), \tag{27}$$

from which it follows that

$$P^2 = 1. \tag{28}$$

The operator P is thus unitary (since it is Hermitian and its own inverse) and it effects a canonical transformation of the $\vec{\tau}^{(1)}$, $\vec{\tau}^{(2)}$. The result of this transformation is, as a simple calculation shows,

$$\begin{aligned} P\vec{\tau}^{(1)}P &= \vec{\tau}^{(2)} \\ P\vec{\tau}^{(2)}P &= \vec{\tau}^{(1)}, \end{aligned} \tag{29}$$

i.e. an interchange of the dichotomic variables of the two systems (1), (2); P is often called the *exchange* operator for the variables τ.

According to (28), P has two distinct eigenvalues $+1$, -1. Eigenfunctions of the eigenvalues $+1$ are symmetrical in $\tau_3^{(1)\prime}$, $\tau_3^{(2)\prime}$, eigenfunctions of -1 antisymmetrical in these arguments; further, they may, of course, be taken as bilinear combinations of the eigenfunctions $\psi_+^{(1)}$, $\psi_-^{(1)}$ of $\tau_3^{(1)}$ and $\psi_+^{(2)}$, $\psi_-^{(2)}$ of $\tau_3^{(2)}$. The three distinct symmetrical combinations

$$\psi_+^{(1)}\psi_+^{(2)} \quad , \quad \psi_-^{(1)}\psi_-^{(2)} \quad , \quad \psi_+^{(1)}\psi_-^{(2)} + \psi_-^{(1)}\psi_+^{(2)} \tag{30}$$

imply a threefold degeneracy of the eigenvalue $+1$; the eigenvalue -1 is simple, with the antisymmetrical eigenfunction

$$\psi_+^{(1)}\psi_-^{(2)} - \psi_-^{(1)}\psi_+^{(2)}. \tag{31}$$

4.14. *Equivalent formulations of the exclusion principle.* When the specification of a particle involves a dichotomic variable, we are at liberty

to treat a system of such particles as a mixture of two distinct kinds of particles, characterized by the two eigenvalues of the dichotomic variables. Thus a system of particles of spin $\frac{1}{2}$ can be regarded as composed of particles of "left" spin and particles of "right" spin; a system of nucleons as a system of protons and neutrons. In the original conception, each particle is defined by an eigenvalue τ_3' of the dichotomic variable and a set Q of other coordinates; in the new conception, we have two distinct sets of coordinates Q.

The exclusion principle can accordingly be formulated in two equivalent ways, by stating either that any wave-function of the arguments $Q^{(i)}$, $\tau_3^{(i)'}$ must be antisymmetrical in *all* coordinates, or that any wave-function of the arguments $Q^{(i_1)}$, $Q^{(i_2)}$ pertaining to the two different kinds of particles must be antisymmetrical *separately* with respect to the $Q^{(i_1)}$ and the $Q^{(i_2)}$. The equivalence of these two formulations may readily be seen if one observes that the most general wave-function of a system of A particles, satisfying the exclusion principle as first enunciated, may be expanded in terms of a complete orthogonal system of "Slater-determinants" *, constructed in the following way. Take any operator (such as the total kinetic energy of the system) which depends additively on the individual particles and does not involve any coupling between the dichotomic variable and the other coordinates; the eigenfunctions of the corresponding operator for a single nucleon will then be of the form $\psi_n(Q)u_\pm(\tau)$, where τ is put for the eigenvalue τ_3' of the dichotomic variable and $u_\pm(\tau)$ are the eigenfunctions of τ_3:

$$\begin{aligned} u_+(+1) &= u_-(-1) = 1 \\ u_+(-1) &= u_-(+1) = 0; \end{aligned} \tag{32}$$

and the eigenfunctions of the total operator, satisfying the exclusion principle, may be written as determinants of the type

$$\begin{vmatrix} \psi_{n_1}(Q^{(1)})\,u_+(\tau^{(1)}) & \cdots & \psi_{n_1}(Q^{(A)})\,u_+(\tau^{(A)}) \\ \cdots\cdots & & \cdots\cdots \\ \psi_{n_N}(Q^{(1)})\,u_+(\tau^{(1)}) & \cdots & \psi_{n_N}(Q^{(A)})\,u_+(\tau^{(A)}) \\ \psi_{n_{N+1}}(Q^{(1)})\,u_-(\tau^{(1)}) & \cdots & \psi_{n_{N+1}}(Q^{(A)})\,u_-(\tau^{(A)}) \\ \cdots\cdots & & \cdots\cdots \\ \psi_{n_A}(Q^{(1)})\,u_-(\tau^{(1)}) & \cdots & \psi_{n_A}(Q^{(A)})\,u_-(\tau^{(A)}) \end{vmatrix}, \tag{33}$$

the upper indices pertaining to the different particles, while $n_1, \ldots, n_A$ represent a permutation of the sequence $1, 2, \ldots, A$. The Slater-determinant (33) refers to N particles in "$+1$" states and $Z = A - N$ particles in "-1" states; on account of (32), it is immediately apparent that it is different from zero only when N of the τ's take the value $+1$ and the Z

* See, e.g., KRAMERS [38], § 71.

other the value -1, in which case it reduces to the form

$$\pm \begin{vmatrix} \psi_{n_1}(Q^{(m_1)}) \dots \psi_{n_1}(Q^{(m_N)}) \\ \dots \\ \psi_{n_N}(Q^{(m_1)}) \dots \psi_{n_N}(Q^{(m_N)}) \end{vmatrix} \cdot \begin{vmatrix} \psi_{n_{N+1}}(Q^{(m_{N+1})}) \dots \psi_{n_{N+1}}(Q^{(m_A)}) \\ \dots \\ \psi_{n_A}(Q^{(m_{N+1})}) \dots \psi_{n_A}(Q^{(m_A)}) \end{vmatrix}, \quad (34)$$

where $m_1, \dots, m_A$ denotes some permutation of $1, 2, \dots, A$. This, now, is just the type of products of Slater-determinants which would have been obtained for the eigenfunctions of a system of N "$+1$" particles and Z "-1" particles, considered as two distinct kinds of particles, each obeying the exclusion principle.

4.141. An alternative proof, due to KLEIN [38], will be given here for the sake of its elegance. We would even have given it in preference, were it not that it is based on the so-called "second quantization" method, the knowledge of which we do not wish to require from the reader. The quantized amplitudes of a system of particles satisfying the exclusion principle in the form stated above are functions of the Q's and τ, say $a(Q, \tau)$, $a^\dagger(Q, \tau)$, which obey the commutation rules

$$\begin{aligned} &[a(Q_1, \tau_1),\ a(Q_2, \tau_2)]_+ = [a^\dagger(Q_1, \tau_1),\ a^\dagger(Q_2, \tau_2)]_+ = 0 \\ &[a(Q_1, \tau_1),\ a^\dagger(Q_2, \tau_2)]_+ = \delta(Q_1, Q_2)\, \delta_{\tau_1 \tau_2}, \end{aligned} \quad (35)$$

where $\delta(Q_1, Q_2)$ is the ordinary or the DIRAC δ-function, according as the Q's have a discrete or continuous range of values. We recall that the product $a^\dagger(Q, \tau) a(Q, \tau) \equiv N(Q, \tau)$ represents the number (1 or 0) of particles in the state (Q, τ); as may readily be deduced from (35), the N's commute with all a's and $a^\dagger$'s, except that

$$a(Q, \tau)\, N(Q, \tau) = [1 - N(Q, \tau)]\, a(Q, \tau). \quad (36)$$

Now, we intend to show the equivalence of this mode of description of the system with another, characterized by two sets of amplitudes $a_+(Q)$, $a_-(Q)$ satisfying the following requirements:

(1) the expressions $a_+^\dagger a_+$, $a_-^\dagger a_-$ represent the numbers of "$+1$" particles and "-1" particles in the state Q, respectively;

(2) the $(a_+, a_+^\dagger)$, $(a_-, a_-^\dagger)$ satisfy separately commutation rules of the same type as (35);

(3) any of the $(a_+, a_+^\dagger)$ commutes with any of the $(a_-, a_-^\dagger)$.

The two first conditions are obviously satisfied by identifying a_+, a_- with $\pm a(\tau = +1)$, $\pm a(\tau = -1)$, respectively; the third condition can then be fulfilled by suitably fixing the double sign. It suffices to put

$$\begin{aligned} a_+(Q) &= a(Q, +1) \\ a_-(Q) &= (-1)^{\sum_Q N(Q,+1)} \cdot a(Q, -1). \end{aligned} \quad (37)$$

From (36), it follows, in fact, that

$$[a(Q, +1),\ (-1)^{N(Q,+1)}]_+ = 0, \quad (38)$$

whence, taking account of (35),

$$\begin{aligned} &[a(Q, +1),\ (-1)^{\sum N(Q,+1)}]_+ = 0 \\ &[a(Q, -1),\ (-1)^{\sum N(Q,+1)}]_- = 0; \end{aligned} \quad (39)$$

from (39) and (35), the relations

$$[a_+(Q_1), a_-(Q_2)]_- = [a_+(Q_1), a_-^+(Q_2)]_- = 0 \tag{40}$$

are immediately deduced, on account of the definition (37).

4.2. Wave-equation of the nucleon

4.21. *Linearization of wave-equation.* A very remarkable use of dichotomic variables has, as is well-known, been made by DIRAC [35] in order to "linearize" the fundamental equation

$$E^2 = M^2 + p^2 \tag{1}$$

between the energy E, rest mass M and momentum p (multiplied by c) of a free particle of spin $\frac{1}{2}$, i.e. to find a rational function of M and of the components of p, the square of which would be equal to the right-hand side of (1). From (*4.11*–1) and (*4.11*–8) we first deduce, for any vector p with commutable components, and commuting with $\vec{\tau}$, the identity

$$p^2 = (\vec{\tau}\,\vec{p})^2, \tag{2}$$

whence the expression $\vec{\tau}\,\vec{p}$ is seen to yield the desired result as regards p^2. We may also decompose the vector $\vec{p}$ into two perpendicular components $p_{||}, p_\perp$; with the help of the dichotomic variables $\tau_{||}, \tau_\perp$ associated with the directions of these components, we may then write, on account of the invariant form of (2),

$$p^2 = p_{||}^2 + p_\perp^2 = (\tau_{||}\, p_{||} + \tau_\perp\, p_\perp)^2. \tag{3}$$

This gives us a purely algebraical identity, expressing that the square of the rational function $\tau_{||}\, p_{||} + \tau_\perp\, p_\perp$, in which $p_{||}, p_\perp$ are any two commutable quantities, while $\tau_{||}, \tau_\perp$ refer to two perpendicular directions, is just $p_{||}^2 + p_\perp^2$.

Turning now to equation (1), we can effect its linearization by means of two dichotomic variables $\vec{\sigma}, \varrho$; the first of these will allow us, according to (2), to replace p^2 by $(\vec{\sigma}\,\vec{p})^2$, while the second will be used, according to (3), to express the right-hand side of (1) as the square of $\varrho_{||} M + \varrho_\perp (\vec{\sigma}\,\vec{p})$. This last expression Dirac then assumes as the Hamiltonian of the free particle:

$$E = \varrho_{||}\, M + \varrho_\perp (\vec{\sigma}\,\vec{p}). \tag{4}$$

On closer discussion, it turns out that the vector $\vec{\sigma}$ can be interpreted as the spin vector of the particle. For this point, as well as for the further development of the theory, we must refer the reader to Dirac's book. Here we shall only mention that the other dichotomic variables $\varrho_{||}$ and $\varrho_\perp$ also receive simple and important interpretations.

We may first enumerate the four components of the wave-function by the eigenvalues of σ_3 and of $\varrho_{||}$, which thus takes the diagonal form * ϱ_3; for $\varrho_\perp$ we may then conveniently choose the form ϱ_1. The variable $\varrho_{||}$ divides these components into two pairs, which we shall denote by $\psi_+ \equiv \psi(\varrho'_{||} = +1)$ and $\psi_- \equiv \psi(\varrho'_{||} = -1)$, respectively. We now write down the wave-equation for ψ_- separately. To find this, first multiply (4) on the right by $\mathsf{P}_- = \frac{1}{2}(1 - \varrho_{||})$; remembering the property of the operator $\varrho_\perp \mathsf{P}_-$, discussed above (*4.12*), of transforming ψ_- into ψ_+, we then get

$$(E + M)\,\psi_- = (\vec{\sigma}\vec{p})\,\psi_+, \tag{5}$$

whence, for a state of given momentum $\vec{p}$ and *positive* energy E, we find for the order of magnitude of the ratio of the amplitudes of ψ_- and ψ_+

$$\left|\frac{\psi_-}{\psi_+}\right| \approx \frac{p}{E + M} \approx \frac{v}{c}, \tag{6}$$

v being the velocity of the electron in the state considered. In view of the limiting case of slow velocities, the pairs of components ψ_+, ψ_- are therefore usually called the "large" and the "small" components, respectively. We thus see that $\varrho_{||}$ discriminates between large and small components of the wave-function.

If, on the other hand, we choose $\varrho_\perp$ as dichotomic variable to enumerate the components of the wave-function instead of $\varrho_{||}$ (so that now $\varrho_\perp = \varrho_3$ and $\varrho_{||} = \varrho_1$, say), we get another separation into two pairs, called "spinors", each of which transforms according to a two-dimensional irreducible representation of the Lorentz group (excluding reflections). We shall, however, not further discuss this aspect of the theory, of which no use is made in the book.

4.22. *Isotopic variable.* In conformity with the conception of proton and neutron as states of different charge of the nucleon, we introduce a dichotomic variable to distinguish these states. It will be called the *isotopic variable* **, we make the convention that its eigenvalues $+1$, -1 denote neutron and proton states, respectively. To the isotopic variable corresponds, according to the general theory, a fundamental vector in a symbolical space, which will be denoted (to distinguish it from vectors in ordinary space) by a clarendon type $\boldsymbol{\tau}$, while its components τ_1, τ_2, τ_3, as defined by (*4.11*–6), will be distinguished by clarendon type indices. The corresponding symbolical space will be called the *isotopic space*. The wave-function of the nucleon will then be considered as depending also on the eigenvalues τ'_3 of the isotopic variable τ_3. With the help of the operators

$$\mathsf{T}_+ = \tfrac{1}{2}(1 + \tau_3) \quad , \quad \mathsf{T}_- = \tfrac{1}{2}(1 - \tau_3), \tag{7}$$

* Dirac takes $\varrho_{||} = -\varrho_3$, $\varrho_\perp = -\varrho_1$, which only amounts to a permutation of the respective roles of the eigenvalues $+1$ and -1 of $\varrho_{||}$.

** The usual name is "isotopic spin".

the mass M of the nucleon is expressed as an operator

$$M = M_n \mathsf{T}_+ + M_p \mathsf{T}_- . \tag{8}$$

If we adopt (*1.21*) for the nucleon the linearization procedure (*4.21*) of Dirac, the Hamiltonian of a free nucleon takes the form (4), in which M has to be replaced, strictly speaking, by its expression (8); it will, however, mostly be permissible to neglect the effects of the mass difference between proton and neutron and treat M as a constant *.

4.221. *Interaction with electromagnetic field.* In order to account for the interaction of a nucleon with an external electromagnetic field, we must add to the Hamiltonian not only the usual terms connected with the charge of the proton, but also other terms arising from the anomalous magnetic moment (*1.21, 1.22*). The Hamiltonian representing this interaction correctly to the first order of approximation in the nucleon velocity is the following:

$$\begin{aligned} \mathcal{H} = \varrho_1 \vec{\sigma}(\vec{p} - \mathsf{T}_- e\vec{A}) + \mathsf{T}_- e V + \varrho_3 M \\ - [\mu_n \mathsf{T}_+ + (\mu_p - 1)\mathsf{T}_-] \mu_0 \vec{\sigma}\vec{H}, \end{aligned} \tag{9}$$

$\vec{A}$, V denoting the electromagnetic (vector and scalar) potential, and $\vec{H}$ the magnetic field, taken at the point $\vec{x}$ occupied by the nucleon. From this Hamiltonian the wave-equation is derived by interpreting the operators $\mathcal{H}$, $\vec{p}$ as

$$\mathcal{H} \rightarrow \frac{i\hbar}{c}\frac{\partial}{\partial t} \quad , \quad \vec{p} \rightarrow \frac{\hbar}{i}\,\mathrm{grad}. \tag{10}$$

More rigorously, we should include into the Hamiltonian the interaction of the nucleon with the nuclear field (*1.21*), which should automatically give rise, in first approximation, to the supplementary terms of equation (9). In this way, the magnetic anomalies are related to the property of nucleons of going over from neutron to proton states, and conversely, with emission of mesons or leptons; this may certainly be regarded as an attractive feature of the field conception. From a formal point of view, considered as a method for correcting Dirac's equation, when applied to nucleons, without upsetting its main properties, such a procedure, however, is admittedly wrong in so far as it partakes of all the evils of quantum field theory, and it remains to be seen whether a much more radical departure from the present formalism will not prove unavoidable.

* From the point of view of a fundamental theory of elementary particles, one has to keep in mind the possibility that the mass parameter occurring in the wave equation of a nucleon should actually be a constant M, irrespective of the charge of the nucleon, while the difference between the observed masses of proton and neutron should be attributed to the energy of the proper electromagnetic and nuclear fields created by these particles.

4.3. Reduction of the wave-equation of the two-nucleon system

4.31. *The fundamental wave-equation.* The wave-equation of the two-nucleon system can be derived, to the first order in the nucleon velocities, from the Hamiltonian

$$H' = \varrho_1^{(1)} \vec{\sigma}^{(1)} \vec{p}^{(1)} + \varrho_1^{(2)} \vec{\sigma}^{(2)} \vec{p}^{(2)} + (\varrho_3^{(1)} + \varrho_3^{(2)}) M + V, \tag{1}$$

in which the nucleon mass M is considered as a constant and the interaction operator V includes the Coulomb energy

$$\frac{e^2}{r} \underline{\tau}^{(1)} \underline{\tau}^{(2)}.$$

Strictly speaking, a Hamiltonian of the form (1) is Lorentz invariant only when the potential energy V corresponds to a *"contact" interaction*, i.e. is proportional to $\delta(\vec{x}) \equiv \delta(x)\delta(y)\delta(z)$. In fact, this mode of interaction is the only one allowed by relativity requirements as long as one has to do with a *direct* interaction, without any intermediary field. For the only admissible expressions of the interaction energy, from this point of view, are of the type

$$\int s_{ijk\ldots}(\vec{x})\, s^{ijk\ldots}(\vec{x})\, dv,$$

involving products of source densities $s_{ijk\ldots}$ taken at the same point of space (and the same time); this just corresponds to an interaction potential of the form $\delta(\vec{x})$. Still, the Hamiltonian (1) may be used with any other form of distance dependence of V up to the first order of approximation in the nucleon velocities. This will be shown in a later section (*15.22*), in which second order corrections will also be summarily discussed.

The operator V, which is symmetrical in the sets of coordinates of the two interacting nucleons, has to satisfy the following general requirements of invariance:

(*a*) conservation of energy demands that it be Hermitian;

(*b*) conservation of momentum and Galilean relativity, that it depend on the position and momentum variables of the nucleon pair only through the relative coordinates and momenta *;

(*c*) conservation of angular momentum, that it be invariant under spatial rotations;

(*d*) V must also be invariant under spatial reflections (this property does not correspond to any classical conservation law **);

---

* Conservation of momentum means invariance with respect to translations, the infinitesimal transformation operators of which are proportional to the components of the total momentum. The operators of infinitesimal Galilei transformations are proportional to the coordinates of the centre of gravity.

** One might speak of "conservation of parity" (in a statistical sense); cf. E. WIGNER, *Gött. Nachr.* 1927, p. 375. The concept of *parity* is defined in *4.331*.

(*e*) invariance under a change of sign of the time variable implies that $\mathcal{V}$ remains invariant when the momenta and spin variables change sign, while at the same time any operator $\Pi = \frac{1}{2}(\tau_1 - i\tau_2)$, effecting a neutron-proton transition (*4.12–24*), is replaced by the corresponding $\Pi^\dagger$, which effects the reverse transition, and vice versa;

(*f*) conservation of charge, that $\mathcal{V}$ commute with $\tau_3^{(1)} + \tau_3^{(2)}$, in other words that it be invariant under rotations about the 3-axis in symbolical isotopic space.

In addition, we shall now assume that

(*g*) $\mathcal{V}$ does not contain either the momenta $\vec{p}^{(1)}$, $\vec{p}^{(2)}$ or the variables $\varrho_i^{(1)}$, $\varrho_i^{(2)}$ $(i = 1, 2, 3)$;

this means that we have to do with a *static interaction*. This is the essential restriction underlying the following developments. Combining it with the invariance requirements (*b*), (*c*) and (*d*) above, we conclude that

(*h*) $\mathcal{V}$ is symmetric with respect to the spin variables $\vec{\sigma}^{(1)}$, $\vec{\sigma}^{(2)}$.

4.311. *Reduction to the barycentric system of reference.* Let us introduce the coordinates

$$\vec{x}_b = \frac{\vec{x}^{(1)} + \vec{x}^{(2)}}{2} \quad , \quad \vec{p}_b = \vec{p}^{(1)} + \vec{p}^{(2)} \tag{2}$$

of the centre of gravity of the system and the relative coordinates

$$\vec{x} = \vec{x}^{(1)} - \vec{x}^{(2)} \quad , \quad \vec{p} = \tfrac{1}{2}(\vec{p}^{(1)} - \vec{p}^{(2)}). \tag{3}$$

Applying to the Hamiltonian (1) the canonical transformation (2), (3), we get, on account of assumption (*b*) about $\mathcal{V}$,

$$\mathcal{H}' = \tfrac{1}{2}(\varrho_1^{(1)}\vec{\sigma}^{(1)} + \varrho_1^{(2)}\vec{\sigma}^{(2)})\vec{p}_b + \mathcal{H}, \tag{4}$$

with

$$\mathcal{H} = (\varrho_1^{(1)}\vec{\sigma}^{(1)} - \varrho_1^{(2)}\vec{\sigma}^{(2)})\vec{p} + (\varrho_3^{(1)} + \varrho_3^{(2)})M + \mathcal{V}. \tag{5}$$

From this it appears that $\vec{p}_b$, commuting with $\mathcal{H}'$, is a constant of the motion. We may therefore restrict ourselves to such states of the system for which

$$\vec{p}_b = 0, \tag{6}$$

i.e. the centre of gravity is at rest. In this *barycentric system of reference*, the Hamiltonian, expressed in the relative coordinates, is simply given by $\mathcal{H}$, as defined by (5). The total angular momentum (multiplied by c) is, in this system,

$$\hbar\vec{J} \equiv \vec{x} \wedge \vec{p} + \frac{\hbar}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}); \tag{7}$$

by assumption (c) about $\mathcal{V}$, its three components are constants of the motion.

4.32. *Classification of eigenstates by angular momentum.* In the usual way, let us consider the commutable operators J^2 and J_z (z denoting any fixed direction in space): the eigenfunctions of the Hamiltonian $\mathcal{H}$ will be linear combinations of the simultaneous eigenfunctions of these two operators. The latter may be obtained as follows. For the square of the spin angular momentum (in units $\hbar$), we get in the first place

$$\tfrac{1}{4}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})^2 = \tfrac{1}{2}(3 + \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}) = P_\sigma + 1, \tag{8}$$

P_σ being the exchange operator (*4.13*) for the spin variables. The spin eigenstates therefore consist of two systems, corresponding to the eigenvalues 0 and 2 of the quantity (8): a non-degenerate or *singlet* system and a trebly degenerate or *triplet* system. Their respective eigenfunctions may be written, if v_+, v_- denote in the usual way the eigenfunctions of σ_z with respective eigenvalues $+1$, -1:

$$\begin{aligned}
\textit{Singlet:} \quad & {}^1(\sigma)_0 = \frac{1}{\sqrt{2}}[v_+(1)\,v_-(2) - v_-(1)\,v_+(2)] \\
\textit{Triplet:} \quad & {}^3(\sigma)_0 = \frac{1}{\sqrt{2}}[v_+(1)\,v_-(2) + v_-(1)\,v_+(2)] \\
& {}^3(\sigma)_1 = v_+(1)\,v_+(2) \\
& {}^3(\sigma)_{-1} = v_-(1)\,v_-(2);
\end{aligned} \tag{9}$$

these are at the same time eigenfunctions of the z-component of the spin $\frac{1}{2}(\sigma_z^{(1)} + \sigma_z^{(2)})$, corresponding to the eigenvalues given by the lower indices of the symbols (σ). It is easy to see that for the singlet states, we actually get

$$\tfrac{1}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\,{}^1(\sigma)_0 = 0 \tag{10}$$

for all spin components.

On the other hand, the square and the z-component of the orbital angular momentum (in units $\hbar$)

$$\vec{L} = \frac{1}{\hbar}\vec{x} \wedge \vec{p} \tag{11}$$

have as eigenfunctions the tesseral harmonics * $Y_l^m(\vartheta, \varphi)$, corresponding to the eigenvalues $l(l+1)$ of L^2 and m of L_z. Since, by (7), (8),

$$J^2 = L^2 + P_\sigma + 1 + (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\vec{L} \tag{12}$$

and

$$J_z = L_z + \tfrac{1}{2}(\sigma_z^{(1)} + \sigma_z^{(2)}), \tag{13}$$

* The definitions and notations adopted for the functions are explained in *N. 14.* Here, r, ϑ, φ are the polar coordinates of particle (1) with respect to particle (2), the polar axis being in the z-direction.

we see directly that for the singlet system, the eigenvalues of J^2 and J_z are, on account of (10), just the same as those of L^2 and L_z, so that the eigenfunctions are simply

$$ {}^1Z_l^{(l)m} = {}^1(\sigma)_0\, Y_l^m . \tag{14} $$

For the triplet system, we may, according to (13), write the eigenfunctions corresponding to a given value l of the orbital angular momentum and to a given eigenvalue m of J_z in the form

$$ {}^3Z_j^{(l)m} = a_j \cdot {}^3(\sigma)_0\, Y_l^m + b_j \cdot {}^3(\sigma)_1\, Y_l^{m-1} + c_j \cdot {}^3(\sigma)_{-1}\, Y_l^{m+1} . \tag{15} $$

The eigenvalues λ_j of J^2 as well as the coefficients a_j, b_j, c_j result, by (12), from the eigenvalue-problem

$$ [l(l+1)+2-\lambda_j] \cdot {}^3Z_j^{(l)m} + (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\, \vec{L} \cdot {}^3Z_j^{(l)m} = 0. \tag{16} $$

Using the well-known formulae for the matrix-elements of $\vec{L}$, this is readily reduced to a set of three homogeneous linear equations in a_j, b_j, c_j. Putting $\lambda_j = j(j+1)$, the zeros of the secular determinant are

$$ j = l \quad \text{and} \quad j = l \pm 1 ; $$

the normalized eigenfunctions of the form (15) are found to be

$$ \begin{aligned} {}^3Z_l^{(l)m} &= \frac{1}{\sqrt{2\,l(l+1)}} [\sqrt{2}\, m \cdot {}^3(\sigma)_0 \cdot Y_l^m \\ &\quad + \sqrt{(l+m)(l-m+1)} \cdot {}^3(\sigma)_1 \cdot Y_l^{m-1} - \sqrt{(l-m)(l+m+1)} \cdot {}^3(\sigma)_{-1} \cdot Y_l^{m+1}] \\ {}^3Z_{l+1}^{(l)m} &= \frac{1}{\sqrt{2\,(l+1)(2\,l+1)}} [\sqrt{2}\, \sqrt{(l+m+1)(l-m+1)} \cdot {}^3(\sigma)_0 \cdot Y_l^m \\ &\quad - \sqrt{(l+m)(l+m+1)} \cdot {}^3(\sigma)_1 \cdot Y_l^{m-1} - \sqrt{(l-m)(l-m+1)} \cdot {}^3(\sigma)_{-1} \cdot Y_l^{m+1}] \\ {}^3Z_{l-1}^{(l)m} &= \frac{1}{\sqrt{2\,l(2\,l+1)}} [\sqrt{2}\, \sqrt{(l+m)(l-m)} \cdot {}^3(\sigma)_0 \cdot Y_l^m \\ &\quad + \sqrt{(l-m)(l-m+1)} \cdot {}^3(\sigma)_1 \cdot Y_l^{m-1} + \sqrt{(l+m)(l+m+1)} \cdot {}^3(\sigma)_{-1} \cdot Y_l^{m+1}]. \end{aligned} \tag{17} $$

4.33. *Reduction to "large" components.* The distribution between "large" and "small" components of the wave-functions (*4.21*) is effected by the operator $\frac{1}{2}(\varrho_3^{(1)} + \varrho_3^{(2)})$, the eigenfunctions of which form a set

$$ {}^1(\varrho)_0;\ {}^3(\varrho)_0,\ {}^3(\varrho)_1,\ {}^3(\varrho)_{-1} $$

built up in a way entirely analogous to (9) with the eigenfunctions w_+, w_- of the dichotomic variable ϱ_3. Any wave-function of our problem may be written as

$$ \begin{aligned} \psi &= \psi^{(0)} + \psi^{(1)} + \psi^{(2)}, \\ \psi^{(0)} &= {}^3f_1 \cdot {}^3(\varrho)_1 \\ \psi^{(1)} &= {}^3f_0 \cdot {}^3(\varrho)_0 + {}^1f_0 \cdot {}^1(\varrho)_0 \\ \psi^{(2)} &= {}^3f_{-1} \cdot {}^3(\varrho)_{-1} ; \end{aligned} \tag{18} $$

$\psi^{(0)}$ is then the "large" component, $\psi^{(1)}$ is of the first order in the nucleon velocities, $\psi^{(2)}$ of the second order in these velocities. In the following we shall, in conformity with our initial assumptions (*4.31*), perform the reduction of the wave-equation only to the first order of approximation in the nucleon velocities; the extension of the method to a higher degree of accuracy would, however, be quite straightforward *.

Inserting (18) into the equation $\mathcal{H}\psi = E\psi$, taking account of assumption (g) about $\mathcal{V}$, and observing that the "non-relativistic" energy

$$\varepsilon = E - 2M \tag{19}$$

and $\mathcal{V}$ are also negligible in comparison with $2M$, we get

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})\vec{p} \cdot {}^3f_0 - \frac{1}{\sqrt{2}}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\vec{p} \cdot {}^1f_0 + \mathcal{V} \cdot {}^3f_1 = \varepsilon \cdot {}^3f_1 \\
&\frac{1}{\sqrt{2}}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})\vec{p} \cdot {}^3f_1 = 2M \cdot {}^3f_0 \\
&-\frac{1}{\sqrt{2}}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\vec{p} \cdot {}^3f_1 = 2M \cdot {}^1f_0,
\end{aligned} \tag{20}$$

whence, by eliminating ${}^3f_0$ and ${}^1f_0$ and using (*4.11–9*), the wave-equation for the large component

$$\left(\frac{p^2}{M} + \mathcal{V}\right) \cdot {}^3f_1 = \varepsilon \cdot {}^3f_1 \tag{21}$$

follows in the form of the ordinary Schrödinger equation with the reduced mass $\frac{1}{2}M$. Once ${}^3f_1$ is determined, the small components ${}^3f_0$ and ${}^1f_0$ may be calculated from the last two relations (20).

4.331. *Properties of the large components.* Owing to the property (h) of $\mathcal{V}$ and to formula (8), the potential commutes with the square of the total spin and thus conserves the multiplicity of the large components ${}^3f_1$: these eigenfunctions separate into two distinct groups, the *singlet* and the *triplet systems*. Likewise, on account of the rotation invariance (c) of $\mathcal{V}$, the functions ${}^3f_1$ correspond to definite values of the quantum numbers j and m. In the singlet system, the orbital momentum is hereby also fixed ($l = j$). But in the triplet system, it will in general not have a definite eigenvalue: the functions ${}^3f_1$ will be linear combinations of the ${}^3Z_j^{(l)m}$ with $|j-1| \leq l \leq j+1$.

The character of this "mixture" of states of different orbital momenta can be more closely determined by taking into consideration the *parity* of the eigenstates. When a spatial symmetry transformation with respect to the origin is performed, the eigenfunction may either resume its original value or change sign: in the former case, we have to do with an *even*

* A rigorous treatment, involving the use of group theory, is due to KEMMER [37a]. The discussion given here is quite elementary.

state; in the latter, with an *odd* state. This transformation property is called the *parity* of the state; it can be designated either by the words "even", "odd" or by the corresponding eigenvalues $+1$, -1 of the spatial symmetry operator. Any state of orbital momentum l has the parity $(-1)^l$. Now, by property (d) of the potential, the large components ${}^3f_1$ must also have a definite parity; they can therefore only involve states with either even or odd values of l. Of the three possible l-values for a given j, viz. $|j-1|$, j, $j+1$, the two extreme ones are of the same parity, while the middle one is of opposite parity. For every value of j, there are thus two distinct triplet states, the one of which has a definite orbital momentum $l=j$, while the other is a mixture of states of orbital momenta $l=j\pm 1$. Using the well-known spectroscopic notation ${}^\tau L_j$ for states of definite orbital momentum, we may express the result of this discussion by the following table:

j	Even states	Odd states
0		${}^3P_0$
1	${}^3S_1+{}^3D_1$	${}^3P_1$
2	${}^3D_2$	${}^3P_2+{}^3F_2$
3	${}^3D_3+{}^3G_3$	${}^3F_3$
	etc.	

(22)

The coefficients of the Z's in the ${}^3f_1$ may further be decomposed into a radial factor and an eigenfunction of the total isotopic variable $\frac{1}{2}(\tau_3^{(1)}+\tau_3^{(2)})$, characterizing the given total charge of the system. In a notation entirely analogous to (9), the latter eigenfunctions form a set

$$ {}^\tau(\tau)_{m_t}: \quad {}^1(\tau)_0 \; ; \; {}^3(\tau)_{-1} \; , \; {}^3(\tau)_0 \; , \; {}^3(\tau)_{+1} $$

in which the v's occurring in (9) are replaced by the eigenfunctions u_+, u_- of τ_3. The eigenvalue m_t of the total isotopic variable is also a quantum number of the eigenstate; the values $m_t=-1$, 0, $+1$ refer to the two-proton, proton-neutron and two-neutron system, respectively.

The symmetry of the charge eigenfunction is indicated by $\tau=1$ or 3 ("charge singlet", "charge triplet"). The particular eigenfunction ${}^\tau(\tau)_{m_t}$ occurring in the coefficient of a given Z is uniquely fixed by the exclusion principle in the following way. When space and spin coordinates of the two particles are interchanged, the functions ${}^1Z_l^{(l)m}$, ${}^3Z_j^{(l)m}$ are multiplied by $(-1)^{l+1}$ and $(-1)^l$, respectively. In order that the total eigenfunction be antisymmetrical when we also interchange the isotopic variables, we must therefore take:

	Even states	Odd states
Singlet	${}^3(\tau)$	${}^1(\tau)$
Triplet	${}^1(\tau)$	${}^3(\tau)$

(23)

For a system of two protons or two neutrons, the eigenfunction ${}^{\tau}(\tau)_{m_t}$ has necessarily the form ${}^{3}(\tau)_{\pm 1}$ and the possible eigenstates are accordingly restricted to the "charge symmetrical" states. The proton-neutron system, on the other hand, possesses the full set of possible eigenstates, with the isotopic eigenfunctions ${}^{3}(\tau)_0$ and ${}^{1}(\tau)_0$.

Summing up, the "large" eigenfunctions may thus be written in the general form

$$ {}^{3}f_1 = \sum_{l'} \frac{{}^{\sigma}R^{(l')}_{j\,m_t}(r)}{r} \cdot {}^{\sigma}Z^{(l')\,m}_{j} \cdot {}^{\tau'}(\tau)_{m_t}, \tag{24} $$

the summation over l' reducing to a single term $l' = j$ or extending over $l' = j \pm 1$ according to the above rules; the value of τ' is fixed in terms of σ and the parity of the state by (23). Inserting this in equation (21), it is easily deduced that the radial functions $R(r)$ satisfy the set of differential equations *

$$ -\frac{\hbar^2}{M}\left[\frac{d^2}{dr^2}\,{}^{\sigma}R^{(l)} - \frac{l(l+1)}{r^2}\,{}^{\sigma}R^{(l)}\right] + \sum_{l'}(l\,|\,{}^{\sigma}\mathcal{V}\,|\,l')\cdot{}^{\sigma}R^{(l')} = {}^{\sigma}\varepsilon\cdot{}^{\sigma}R^{(l)}, \tag{25} $$

in which the matrix-elements of $\mathcal{V}$ are defined by

$$ (l\,|\,{}^{\sigma}\mathcal{V}_{j,m_t}\,|\,l') = \sum_{\sigma_z,\tau_3}{}' \int d\Omega\; {}^{\sigma}Z^{(l)\,m*}_{j}\cdot{}^{\tau}(\tau)_{m_t}\cdot\mathcal{V}\cdot{}^{\sigma}Z^{(l')\,m}_{j}\cdot{}^{\tau}(\tau)_{m_t}; \tag{26} $$

the summations are to be performed over the isotopic and spin variables $\tau_3^{(1)'}, \tau_3^{(2)'}; \sigma_z^{(1)'}, \sigma_z^{(2)'}$, the integration over the angle variables ϑ, φ; due to the rotation invariance of $\mathcal{V}$, the quantities (26) are independent of the quantum number m; the charge multiplicity τ is the same for l and l', since these numbers have the same parity.

4.34. *Central potential.* A considerable simplification occurs if the interaction operator $\mathcal{V}$ does not depend on the angle variables. In this case, it has only diagonal matrix elements of the type (26) and the differential system (25) always reduces to separate differential equations for the ${}^{\sigma}R^{(l)}$. This means that the large eigenfunctions ${}^{3}f_1$ all belong to definite orbital quantum numbers. The diagonal matrix element of the central potential $\mathcal{V}$ is, of course, independent of j; in fact, it depends only on the parity of the state, not on the value of l. It is therefore a function ${}^{\sigma}\mathcal{V}^{(l)}_{m_t}$ of r, which may be regarded as the *effective central potential* in the states of the system with given charge, multiplicity and parity. On account of (23), there are, for the proton-neutron system, four independent effective potentials, pertaining to even or odd states of the singlet or triplet system; we may denote these types of states by the symbols ${}^{1}S$, ${}^{1}P$, ${}^{3}S$, ${}^{3}P$ The system of two like nucleons, on the other hand, has only eigenstates

* Indices j, m_t affecting the symbols $R, \mathcal{V}, \varepsilon$ have been omitted from formula (25).

of the types ^1S and ^3P and accordingly two distinct effective potentials.

The radial equation, of which ${}^\tau R^{(l)}$ is a proper solution, has the form *

$$\left\{-\frac{\hbar^2}{M}\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]+{}^\sigma V^{(l)}(r)\right\}{}^\tau R^{(l)}={}^\sigma\varepsilon^{(l)}\cdot{}^\tau R^{(l)}, \tag{27}$$

and the large component of the eigenfunction may be written

$${}^3f_1=\frac{{}^\tau R^{(l)}_{m_t}(r)}{r}\cdot{}^\sigma Z_j^{(l)\,m}(\vartheta,\varphi;\sigma_z^{(1)\prime},\sigma_z^{(2)\prime})\cdot{}^\tau(\tau)_{m_t}, \tag{28}$$

τ being determined by (23).

The calculation of the small components ${}^3f_0$, ${}^1f_0$ by the last two of formulae (20) can be shortened by the observation that the operators on the left-hand side, being rotation invariant, transform any ${}^\sigma Z_j^{(l)\,m}$ into a linear combination of other Z's with the same j and m; in order to determine the coefficients of these linear expressions, it is therefore sufficient to consider only one term of the ${}^3Z$'s as given by (15), e.g. $a_j\cdot{}^3(\sigma)_0\cdot Y_l^m$. Further the operator $(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})\,\vec{p}$ in the equation for ${}^1f_0$ does not alter the multiplicity σ, while that in ${}^3f_0$ exchanges triplet and singlet states; and according to well-known formulae, the effect of the operator $\vec{p}$ is to transform a $Z^{(l)}$ into a combination $\alpha Z^{(l-1)}+\beta Z^{(l+1)}$. For the singlet system, we thus get

$${}^3f_0\sim\alpha\cdot{}^3Z_l^{(l-1)\,m}+\beta\cdot{}^3Z_l^{(l+1)\,m}$$

and, by (10),

$${}^1f_0=0;$$

for the triplet system, we may write

$${}^3f_0\sim{}^1Z_l^{(l)\,m}$$

and

$$\text{for } j=l: \qquad {}^1f_0\sim\alpha'\cdot{}^3Z_l^{(l-1)\,m}+\beta'\cdot{}^3Z_l^{(l+1)\,m}$$
$$\text{for } j=l\pm1: \quad {}^1f_0\sim{}^3Z_j^{(j)\,m}.$$

The remaining explicit calculation is very short and finally yields, with (18), the following results;

Singlet system:

$$\overset{(0)}{\psi}={}^\tau(\tau)_{m_t}\cdot{}^3(\varrho)_1\cdot{}^1Z_l^{(l)\,m}\cdot{}^1R^{(l)}_{m_t}(r)/r$$

$$\overset{(1)}{\psi}={}^\tau(\tau)_{m_t}\cdot{}^3(\varrho)_0\left[\sqrt{\frac{l}{2l+1}}\,{}^3Z_l^{(l-1)\,m}\,\mathcal{D}_{-l}+\sqrt{\frac{l+1}{2l+1}}\,{}^3Z_l^{(l+1)\,m}\,\mathcal{D}_{l+1}\right]{}^1R^{(l)}_{m_t} \tag{29a}$$

* The index m_t should be added to the symbols $\mathcal{V}$, R, ε in (27).

Triplet system:

$$\overset{(0)}{\psi} = {}^{\tau}(\tau)_{m_t} \cdot {}^{3}(\varrho)_1 \cdot {}^{3}Z_j^{(l)\,m} \cdot {}^{3}R_{m_t}^{(l)}(r)/r$$

$$l=j: \quad \overset{(1)}{\psi} = {}^{\tau}(\tau)_{m_t} \cdot {}^{1}(\varrho)_0 \left[\sqrt{\frac{j+1}{2j+1}}\, {}^{3}Z_j^{(j-1)\,m}\, \mathcal{D}_{-j} - \sqrt{\frac{j}{2j+1}}\, {}^{3}Z_j^{(j+1)\,m}\, \mathcal{D}_{j+1} \right] {}^{3}R_{m_t}^{(j)}$$

$$l=j-1: \quad \overset{(1)}{\psi} = {}^{\tau}(\tau)_{m_t} \left[{}^{1}(\varrho)_0 \sqrt{\frac{j+1}{2j+1}}\, {}^{3}Z_j^{(j)\,m} + {}^{3}(\varrho)_0 \sqrt{\frac{j}{2j+1}}\, {}^{1}Z_j^{(j)\,m} \right] \mathcal{D}_j\, {}^{3}R_{m_t}^{(j-1)} \tag{29b}$$

$$l=j+1: \quad \overset{(1)}{\psi} = {}^{\tau}(\tau)_{m_t} \left[-{}^{1}(\varrho)_0 \sqrt{\frac{j}{2j+1}}\, {}^{3}Z_j^{(j)\,m} + {}^{3}(\varrho)_0 \sqrt{\frac{j+1}{2j+1}}\, {}^{1}Z_j^{(j)\,m} \right] \mathcal{D}_{-(j+1)}\, {}^{3}R_{m_t}^{(j+1)}.$$

In these formulae, the symbol $\mathcal{D}_j$ denotes the radial operator

$$\mathcal{D}_j \equiv -\frac{i\hbar}{\sqrt{2}M} \cdot \frac{1}{r}\left[\frac{d}{dr} - \frac{j}{r}\right]. \tag{30}$$

4.341. *Exchange potentials.* Besides ordinary central forces, described by a potential operator $J(r)$, other types of central interactions, known as *exchange* interactions, have played a considerable part in the historical development of the study of nuclear forces. Although they are now superseded by more general kinds of nuclear potentials, it will not be superfluous to mention them here for future reference.

Let us first consider a potential operator involving the product of a general central interaction operator and the exchange operator P_x of the position coordinates of the two interacting particles:

$$\mathcal{V}' = \mathcal{V}(r; \sigma, \tau)\, P_x, \tag{31}$$

the function $\mathcal{V}$ depending in an arbitrary way on the spin and charge coordinates $\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}$; $\tau^{(1)}, \tau^{(2)}$. Although the operator P_x affects the angular variables, as it transforms ϑ, φ into $\pi - \vartheta, \varphi + \pi$, the potential $\mathcal{V}'$ can still be regarded as a central potential, because the interchange of position coordinates, on account of the exclusion principle, is equivalent to the interchange of spin and charge coordinates and change of sign of the wave-function; in other words, we may write

$$P_x = -P_\sigma P_\tau. \tag{32}$$

provided the operators are only applied to completely antisymmetrical functions.

As regards the corresponding effective potentials (*4.34*), the effect of the factor P_x, which is obviously the same as the operator of spatial symmetry transformation, is simply to change the sign of the effective potential

for odd states, while it has no effect on the effective potential for even states:

$$^{\sigma}V'^{(l)} = (-1)^{l} \cdot {}^{\sigma}V^{(l)}. \tag{33}$$

Other possibilities are clearly the introduction, as a factor, of one the exchange operators P_σ or P_τ. Also in these cases, analogous properties of the signs of the effective potentials can be enunciated. On the whole, we may distinguish four fundamental types of exchange potentials, respectively defined by

$$\begin{aligned} V_W &= J(r) \quad , \quad V_M = J(r)\, P_x = -J(r)\, P_\sigma P_\tau \\ V_B &= J(r)\, P_\sigma \, , \quad V_H = -J(r)\, P_\tau = J(r)\, P_x P_\sigma \end{aligned} \tag{34}$$

and designated by the names of their promotors: Wigner, Majorana, Bartlett and Heisenberg.

4.4. Electromagnetic interactions

4.41. *The exchange operators.* In a field theory of nuclear forces, the operators of charge and current density of a system of nucleons cannot be expected to be expressed simply by the sum of the corresponding operators pertaining to the individual nucleons. Since at least part of the nuclear field must be assumed to carry electric charge (*1.21*), there will in fact occur virtual transitions (*1.321*) involving the exchange of an elementary charge between a proton and a neutron and this will imply a redistribution of the charges and currents of the system, depending on these virtual exchange processes. The corresponding terms in the charge and current density of the system (just as well as those in the energy operator) will therefore involve the characteristic operators $\Pi^{(i)}$, $\Pi^{(i)\dagger}$ effecting the transformation of a nucleon from neutron to proton state or vice versa (*4.12–24*); more precisely, since a pair of nucleons is involved, they may depend on the operators $\Pi^{(i)}\,\Pi^{(k)\dagger}$ and $\Pi^{(k)}\,\Pi^{(i)\dagger}$, or on the Hermitian combinations

$$\begin{aligned} T^{(ik)} &= \Pi^{(i)}\,\Pi^{(k)\dagger} + \Pi^{(i)\dagger}\,\Pi^{(k)} \\ &= \tfrac{1}{2}\,(\tau_1^{(i)}\,\tau_1^{(k)} + \tau_2^{(i)}\,\tau_2^{(k)}), \end{aligned} \tag{1}$$

$$\begin{aligned} \Pi^{(ik)} &= \frac{1}{i}\,(\Pi^{(i)}\,\Pi^{(k)\dagger} - \Pi^{(i)\dagger}\,\Pi^{(k)}) \\ &= \tfrac{1}{2}\,(\tau_1^{(i)}\,\tau_2^{(k)} - \tau_2^{(i)}\,\tau_1^{(k)}) = \tfrac{1}{2}\,(\boldsymbol{\tau}^{(i)} \wedge \boldsymbol{\tau}^{(k)})_3. \end{aligned} \tag{2}$$

Now, from the general law of formation of the energy-momentum tensor on the one hand, and the electric charge and current density on the other, from the given Lagrangian of the system (BELINFANTE [39, 40], ROSENFELD [40], MØLLER and ROSENFELD [43]), it appears that the former quantity involves a sum, and the latter a difference of adjoint quantities: the interaction energy will consequently embody exchange terms of the

form $T^{(ik)}w^{(ik)}$, while in the charge and current density the exchange terms will contain factors $\Pi^{(ik)}$.

The operator $T^{(12)}$ has the eigenfunctions ${}^3(\tau)_0$, ${}^1(\tau)_0$ and ${}^3(\tau)_{\pm 1}$ with the respective eigenvalues $+1$, -1 and 0 (double). As regards $\Pi^{(12)}$, it has, of course, the same eigenvalues, but it does not commute with $T^{(12)}$; applied to the eigenfunctions ${}^3(\tau)_{\pm 1}$ of a system of two like nucleons, it also gives, of course, zero, but with respect to the deuteron system, it interchanges charge singlet and charge triplet:

$$\Pi^{(12)}\,{}^3(\tau)_0 = i\,{}^1(\tau)_0 \quad , \quad \Pi^{(12)}\,{}^1(\tau)_0 = -i\,{}^3(\tau)_0. \tag{3}$$

One finds further

$$[T^{(12)}, \tau_3^{(1)}] = -[T^{(12)}, \tau_3^{(2)}] = 2i\,\Pi^{(12)}. \tag{4}$$

4.42. *The exchange moments.* The interaction of a system containing charges and currents with a slowly varying external electromagnetic field $\vec{E}$, $\vec{H}$ can, as is well-known (*A2.11–6*), be expanded in the form

$$\mathcal{V}_{\text{el.magn}} = -\vec{E}\vec{P} - \vec{H}\vec{M} - (Q\ \text{grad})\,\vec{E} - \ldots. \tag{5}$$

in terms of the successive 2^n-pole moments; in formula (5) we have omitted the term involving the total charge and written only those depending on the electric dipole moment $\vec{P}$, the magnetic dipole moment $\vec{M}$ and the electric quadrupole moment represented by the tensor Q. For the deuteron system, we must expect, according to the above considerations, such moments also to involve exchange terms (there will be further terms depending explicitly on the nuclear field variables, but these do not play any role as long as no free mesons take part in the process studied); for instance, we shall have

$$\vec{M} = \vec{M}_{\text{nucl}} + \vec{M}_{\text{exch}} \tag{6}$$

with an exchange term of the form

$$\vec{M}_{\text{exch}} = \Pi^{(12)}\,\vec{m}^{(12)}, \tag{7}$$

while $\vec{M}_{\text{nucl}}$ denotes the sum of the moments of the nucleons; if there is no orbital motion (S-state), we have simply

$$\vec{M}_{\text{nucl}} = \vec{\mu}^{(1)} + \vec{\mu}^{(2)}, \tag{8}$$

with (*1.21, 1.22, 4.22*)

$$\vec{\mu}^{(i)} = \mu_0\,(T_+^{(i)}\,\mu_n + T_-^{(i)}\,\mu_p)\,\vec{\sigma}^{(i)}. \tag{9}$$

The exchange moments give no contribution to the expectation values of these quantities in stationary states of the deuteron, but they may give

rise to transitions between such states under the influence of an external field. Moreover, they will generally contribute to the expectation value of the electromagnetic energy of the system in a stationary state, since this depends quadratically on the moments, and $(\Pi^{(12)})^2 = \frac{1}{2}(1 - \tau_3^{(1)}\tau_3^{(2)})$, i.e. $= 1$ in any state of the deuteron.

4.43. *The electric dipole moment and Siegert's theorem.* The explicit calculation of the exchange moments falls outside the scope of this work; a systematic survey has been given by MØLLER and ROSENFELD [43]. It is found, in particular, that the exchange magnetic moment of the deuteron, defined with respect to the centre of gravity of the system, is due solely to the spin dependent part of the meson field. As to the exchange electric dipole moment, it can be shown for any nuclear system to give rise only to effects of the second order in the nucleon velocities, and it can therefore be neglected. There remains thus just the moment due to the separate protons:

$$\vec{P}_{\text{nucl}} = e \sum_i \mathsf{T}_-^{(i)} \vec{x}^{(i)}, \tag{10}$$

which for the deuteron becomes (the moment being again defined with respect to the centre of gravity)

$$\vec{P}_{\text{nucl}} = -\frac{e}{2} \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} \vec{x}, \tag{11}$$

in terms of the relative coordinates $\vec{x}$ (*4.311–3*).

It is interesting to notice that even the interaction operator $-\vec{E}\vec{P}_{\text{nucl}}$ embodies exchange effects; this may be seen by observing that this operator is equivalent, for the calculation of expectation values in stationary states or probabilities of processes involving conservation of the total energy, to

$$\text{grad } V \cdot \vec{P}_{\text{nucl}} - \vec{A} \cdot \dot{\vec{P}}_{\text{nucl}},$$

where $\vec{A}$, V denotes the electromagnetic potential and the dot signifies derivation with respect to ct. The two operators differ in fact only by a time derivative, which has vanishing matrix elements in the cases mentioned. Now,

$$\dot{\vec{P}}_{\text{nucl}} = e \sum_i \mathsf{T}_-^{(i)} \dot{\vec{x}}^{(i)} + e \sum_i \dot{\mathsf{T}}_-^{(i)} \vec{x}^{(i)};$$

while the first term just represents the ordinary contribution from the total proton current, the second term may be written, in first approximation,

$$-\frac{e}{2} \sum_i \frac{i}{\hbar} [\mathsf{T}^{(12)} w^{(12)}, \tau_3^{(i)}] \vec{x}^{(i)},$$

i.e., by (4),

$$\frac{e}{\hbar} \Pi^{(12)} w^{(12)} (\vec{x}^{(1)} - \vec{x}^{(2)}), \tag{12}$$

which is of the exchange type. The expression (12) exhibits a remarkable connexion between the exchange part of $\dot{\vec{P}}_{\text{nucl}}$ and that of the nuclear potential. This relation, known as Siegert's theorem (SIEGERT [37], LAMB and SCHIFF [38]), could be used, at an early stage of development of the field theory, to write down approximately the exchange part of $\dot{\vec{P}}$ when the exchange nuclear potential was known. A more exact calculation (MØLLER and ROSENFELD [43]) shows that even when effects of the first order in the velocities are taken into account, the exchange part of $\dot{\vec{P}}$ is represented by an expression of the form (12), in which, however, $w^{(12)}$ denotes the *static* exchange potential only. It is this somewhat surprising result which authorizes the above-mentioned conclusion that any explicit exchange term in the electric dipole moment can only yield contributions of higher order in the nucleon velocities.

CHAPTER V

THE STATIONARY STATES OF THE PROTON-NEUTRON SYSTEM

5.0. In the following Chapters, the properties of stationary states of two-nucleon systems will be investigated, starting from the radial equation (*4.34–27*) valid on the assumption of a central interaction. Apart from the state of binding of the deuteron, we shall chiefly be concerned with scattering processes, and the results of the theoretical treatment will be confronted with the experimental data. It will be convenient to examine separately the proton-neutron and the proton-proton systems. In particular, the Coulomb interaction in the latter case entails an essential complication, because (as is well-known) such an interaction influences even the asymptotic behaviour of the radial wave-functions, which determines the features of the scattering process. We therefore begin with the proton-neutron system, in which only the nuclear field is operative.

5.1. Properties of short-range central potentials

5.11. *Types of short-range central potentials.* On account of the limited range of the nuclear potential (*2.1*), we shall have to study the behaviour of the radial wave-equation when the function representing the effective potential energy decreases (in absolute value) very rapidly with increasing distance. Considering the behaviour at small distances, on the other hand, we can at once exclude singularities at the origin stronger than a pole of the second order. For if the coefficient of the unknown function in an equation of the form (*4.34–27*) possessed such a singularity, the equation would have no regular integrals, and thus no eigensolutions *. The case of a pole of the second order does not unrestrictedly admit of eigensolutions; we shall leave it aside **. In the following, we shall thus assume that the potential has at most a simple pole at the origin. We propose to show how a discussion based essentially on these assumptions only, leads to a fairly complete survey of the general behaviour of the eigenfunctions.

As regards the analytical form of the potential, we may in the first place think of simple functions such as that resulting from the meson field

* This is an immediate consequence of the general theory of linear differential equations (W & W, Chapter X, especially *10.3*), valid for any linear equation of the second order such that the coefficient of the first derivative of the unknown has at most a simple pole at the critical point considered. In fact, under such conditions, the indicial equation breaks down.

** See N. F. MOTT and H. S. W. MASSEY, *The theory of atomic collisions* (1933), Chapter II, *3.2*, p. 30.

theory, which involve the product $\varkappa r$ of the distance with some constant, the inverse of which can then be regarded as defining the *range* of the potential. But we cannot exclude the possibility of more than one such constant occurring in the expression of V; if there are, for instance, different kinds of mesons with different masses, we shall meet with nuclear potentials of the form $(a_1 e^{-\varkappa_1 r} + a_2 e^{-\varkappa_2 r} + \ldots)/r$. Besides, there is no *a priori* reason why the effective potentials pertaining to different types of stationary states should not have different ranges. In any case, we can refer all distances to one of the range constants occurring in the expression for the potential; this particular constant we shall denote by $\varkappa$ and, for brevity, call $\varkappa^{-1}$ "the" range of the nuclear force. We then introduce as dimensionless variable

$$\xi \equiv \varkappa r. \tag{1}$$

Simple types of short-range potentials which have been extensively studied are the following (see especially B & B, § 12 and *passim*):

(*a*) the potential "well" $\begin{cases} w = 1 \text{ for } 0 \leqslant \xi \leqslant 1 \\ w = 0 \text{ for } \xi > 1 \end{cases}$ (2)

(*b*) the exponential potential $w = e^{-2\xi}$ (3)

(*c*) the Morse * potential $w = 2e^{-2\xi} - e^{-4\xi}$ (4)

(*d*) the „Gauss" potential $w = e^{-\xi^2}$ (5)

(*e*) the meson potential $w = e^{-\xi}/\xi$. (6)

The last one can, as especially emphasized by HULTHÉN [42*a*] (*5.22, 5.231*) conveniently be approximated by

$$w = \frac{e^{-\xi}}{1 - e^{-\xi}}; \tag{7}$$

also the type

$$w = 1 - \operatorname{tgh}^2 \xi \tag{8}$$

has been considered (HYLLERAAS [37]). In all the preceding formulae, the function $w(\xi)$ has been normalized in such a way that it reduces either to ξ^{-1} or to 1 for $\xi \to 0$. The actual effective potentials are then of the form $V = -Jw(\xi)$; the constant J will be called the *strength* of the corresponding effective potential. (In the most general case, both J and $w(\xi)$ must carry indices σ, l referring to the type of stationary state considered **.)

This convention and terminology has a clear-cut physical meaning only if the potential is attractive (or repulsive) at *all* distances. Still, we shall

* This potential has been introduced and especially studied by Morse and his collaborators; their first paper (MORSE *et al.* [36]) contains the fundamental formulae and graphs.

** In the following, we shall, when convenient without obscurity, omit such indices σ, l.

retain them, for the sake of uniformity, also in the more complicated case of a potential repulsive at small distances and going over into an attraction at larger distances (fig. 2.22); the "strength" J is then negative, but the function $w(\xi)$, positive at $\xi = 0$, changes sign and passes through a minimum at some distance ξ_m, after which it rapidly tends to zero: it is, of course, the (positive) value $J\,w(\xi_m)$ which in such a case is of physical importance, at any rate for states of negative or not too large positive energies (5.133).

In order to write the radial equation (4.34–27) in a dimensionless form, we put

$$\eta = \frac{M}{\hbar^2 \varkappa^2}\,\varepsilon$$

$$B = -\frac{M}{\hbar^2 \varkappa^2}\,\mathcal{V} \tag{9}$$

$$b = +\frac{M}{\hbar^2 \varkappa^2}\,J$$

and accordingly get

$$\frac{d^2}{d\xi^2}\,{}^{\tau}R^{(l)} + \left({}^{\tau}B^{(l)} + \eta - \frac{l(l+1)}{\xi^2}\right){}^{\tau}R^{(l)} = 0. \tag{10}$$

Owing to the minus-sign in (9), a *positive* sign of $B = b\,w(\xi)$ means an *attractive* force. To get an idea of the orders of magnitude involved, we note that the unit of energy introduced in (9) amounts, if $\varkappa^{-1}$ is taken $\approx d$ (2.22), to $\left(\frac{\hbar}{e^2}\right)^2 \frac{m}{M}\,m \approx 5\,\mathrm{MeV}$.

5.111. *Normalization of radial eigenfunctions.* The eigenfunctions of the discrete stationary states will be normalized with respect to the variable ξ according to the condition

$$\int_0^\infty |R|^2\,d\xi = 1; \tag{11}$$

to go over to the usual normalization with respect to r, one has thus to multiply the function $R(\xi)$ by the factor $\sqrt{\varkappa}$. For the eigenfunctions of the continuum, we have the condition of normalization in the reduced energy scale η

$$\int_0^\infty d\xi\,R_\eta \int_{\eta-\Delta}^{\eta+\Delta} R_{\eta'}\,d\eta' = 1; \tag{12}$$

we go over to the normalization in the usual energy scale (and variable r) by means of the factor $\frac{1}{\hbar}\sqrt{\frac{M}{\varkappa}}$. In the latter case, it is often convenient to introduce, instead of the energy ε, the momentum p of the relative motion or, better still, the quantity k representing 2π times the de Broglie wave-number (inverse of the wave-length λ); between these quantities, we have

the relations

$$k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} = \frac{\sqrt{M\varepsilon}}{\hbar} = \sqrt{\eta} \cdot \varkappa. \tag{13}$$

5.12. *Behaviour of radial eigenfunction at small and large distances.* In order that the eigenfunction ψ be finite at the origin, the boundary condition

$$^{\sigma}R^{(l)}(0) = 0 \tag{14}$$

must be imposed on the solutions of the radial equation (10). The behaviour of these solutions for small values of ξ is conditioned by the highest singularity of the coefficient of the unknown; from the general theory, as developed in W & W, *10.32*, it results that the solution which vanishes at $\xi = 0$ behaves $\sim \xi^{l+1}$ for small ξ.

Since the nuclear potential becomes negligible at large ξ, the asymptotic form of the function R will be expressible as a linear combination of the independent solutions of the equation

$$\frac{d^2W}{d\xi^2} + \left(\eta - \frac{l(l+1)}{\xi^2}\right) W = 0. \tag{15}$$

For the latter we can take either the confluent hypergeometric functions (W & W, *16.12*) $W_{0,l+\frac{1}{2}}(2iz)$ and $W_{0,l+\frac{1}{2}}(-2iz)$, with $z \equiv \sqrt{\eta}\,\xi$, or the functions related to those of Bessel (W & W, *17.5*):

$$\begin{aligned}
\sqrt{z}\,J_{l+\frac{1}{2}}(z) &= \frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{2}(l+1)\pi i}\left\{W_{0,l+\frac{1}{2}}(2iz) + (-1)^{l+1} W_{0,l+\frac{1}{2}}(-2iz)\right\} \\
\sqrt{z}\,J_{-(l+\frac{1}{2})}(z) &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}l\pi i}\left\{W_{0,l+\frac{1}{2}}(2iz) + (-1)^{l} W_{0,l+\frac{1}{2}}(-2iz)\right\}.
\end{aligned} \tag{16}$$

For half integer values of the index m, the asymptotic series (W & W, *16.3*) for $W_{0,m}(z)$ reduces to a finite sum:

$$W_{0,l+\frac{1}{2}}(2iz) = e^{-iz}\left[1 + \sum_{n=1}^{l} \frac{\Pi_{\nu=0}^{n-1}\left[l(l+1) - \nu(\nu+1)\right]}{n!\,(2iz)^n}\right]. \tag{17}$$

The asymptotic behaviour of $^{\sigma}R^{(l)}$ will consequently be essentially different according to the sign of η. For negative η, i.e. stationary states of binding, $z = i\sqrt{-\eta}\,\xi$, and on account of (17), the eigensolution, which must vanish at infinity, will decrease exponentially at large distances:

$$^{\sigma}R^{(l)} \simeq e^{-\sqrt{-\eta}\,\xi}. \tag{18}$$

For positive η, i.e. stationary states representing a proton and a neutron in relative motion, the formula

$$^{\sigma}R^{(l)} \simeq \alpha\sqrt{z}\,J_{l+\frac{1}{2}}(z) + \beta\sqrt{z}\,J_{-(l+\frac{1}{2})} \quad , \quad z \equiv \sqrt{\eta}\,\xi \tag{19}$$

can be combined with the following, deduced from (16) and (17):

$$\begin{aligned} \sqrt{z}\, J_{l+\frac{1}{2}}(z) &\simeq \sqrt{\frac{2}{\pi}} \sin\left(z - l\frac{\pi}{2}\right) \\ \sqrt{z}\, J_{-(l+\frac{1}{2})}(z) &\simeq \sqrt{\frac{2}{\pi}} (-1)^l \cos\left(z - \frac{\pi}{2}\right); \end{aligned} \tag{20}$$

this leads to the asymptotic form

$${}^{\sigma}R^{(l)} \simeq \frac{1}{\sqrt{\pi}\sqrt{\eta}} \sin\left(\sqrt{\eta}\,\xi - l\frac{\pi}{2} + {}^{\sigma}\delta^{(l)}\right), \tag{21}$$

characterized by the *phase constant* ${}^{\sigma}\delta^{(l)}$; the factor $(\pi\sqrt{\eta})^{-\frac{1}{2}}$ normalizes ${}^{\sigma}R^{(l)}$ to unity in the reduced energy scale, in conformity with (12) *.

While, according to (17), $W_{0,\,l+\frac{1}{2}}(\pm 2iz) \sim z^{-l}$ for small $|z|$, and (W & W, *17.24*) the same is true of $\sqrt{z}\, J_{-(l+\frac{1}{2})}(z)$, one has $\sqrt{z}J_{l+\frac{1}{2}}(z) \sim z^{l+1}$, so that the latter function represents the solution of equation (15) satisfying the boundary condition (14). In other words, the solution ${}^{\sigma}R^{(l)}$ of our problem (10) for a positive value of η goes over into $\sqrt{z}\, J_{l+\frac{1}{2}}(z)$ when the potential ${}^{\sigma}B^{(l)}$ vanishes. Comparison of the asymptotic form (21) with the first formula (20) illustrates the significance of the phase ${}^{\sigma}\delta^{(l)}$: this constant expresses the influence of the interaction on the behaviour of the moving nucleons at large distances from each other. The importance of the distortion of the wave-function by the nuclear potential will depend on the magnitude of this function in the region where the potential is effective. Considering the behaviour for small values of $\sqrt{\eta}\,\xi$ of the wave-functions of different orbital momenta, we shall therefore, for a given energy, expect the phases to decrease with increasing l. The main effect will always be that on S-waves, while the distortion of the waves of higher orbital momentum will become increasingly important as the energy increases.

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* From equation (10) we derive by partial integration, for two states with the same σ, l

$$\int_0^{\xi_0} R_\eta R_{\eta'}\, d\xi = \left\{ R_\eta \frac{dR_{\eta'}}{d\xi} - R_{\eta'} \frac{dR_\eta}{d\xi} \right\}_{\xi=\xi_0} \cdot \frac{1}{\eta - \eta'},$$

so that the normalization condition (12) takes the form

$$\lim_{\xi_0 \to \infty} \int_{\eta-\Delta}^{\eta+\Delta} \left\{ R_\eta \frac{dR_{\eta'}}{d\xi} - R_{\eta'} \frac{dR_\eta}{d\xi} \right\}_{\xi=\xi_0} \cdot \frac{d\eta'}{\eta - \eta'} = 1,$$

depending only on the asymptotic form of the solution; it is easily verified that (21) satisfies this condition.

5.121. *Some useful identities.* Let ${}^{\sigma}R^{(l)}$ be a proper solution of equation (10) and W any solution of equation (15) for the same values of l and η. The integral $\int_0^\infty W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)} d\xi$ can easily be shown to depend only on the behaviour of W and ${}^{\sigma}R^{(l)}$ for $\xi = 0$ and $\xi = \infty$. In fact, if we take ${}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}$ from equation (10), we get

$$\int_0^\infty W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = -\int_0^\infty W \left[\frac{d^2}{d\xi^2} + \eta - \frac{l(l+1)}{\xi^2}\right] {}^{\sigma}R^{(l)}\, d\xi,$$

whence, by partial integration, and taking account of (15)

$$\int_0^\infty W \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = \left[{}^{\sigma}R^{(l)} \frac{dW}{d\xi} - W \frac{d\,{}^{\sigma}R^{(l)}}{d\xi}\right]_0^\infty. \tag{22}$$

This formula can be specialized in various ways *. For a state of binding ($\eta < 0$), we get, using (17) and (16),

$$\int_0^\infty W_{0,l+\frac{1}{2}}(2\sqrt{-\eta}\,\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = \frac{2l+1}{l+1}\left[\frac{d\,{}^{\sigma}R^{(l)}}{d\xi} \cdot W_{0,l+\frac{1}{2}}(2\sqrt{-\eta}\,\xi)\right]_{\xi=0} \tag{23}$$

$$\int_0^\infty \sqrt{\sqrt{-\eta}\,\xi} \cdot I_{l+\frac{1}{2}}(\sqrt{-\eta}\,\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = \frac{(-1)^{l+1}}{\sqrt{2\pi}} \lim_{\xi\to\infty}\left[{}^{\sigma}R^{(l)} \frac{dW}{d\xi} - W \frac{d\,{}^{\sigma}R^{(l)}}{d\xi}\right]; \tag{24}$$

in (24), we have put $I_n(z) \equiv i^{-n} J_n(iz)$ and $W \equiv W_{0,l+\frac{1}{2}}(-2\sqrt{-\eta}\,\xi)$. For a state of relative motion ($\eta > 0$), (20) and (21) give

$$\frac{\pi}{\sqrt{2}} \int_0^\infty \sqrt{\xi}\, J_{l+\frac{1}{2}}(\sqrt{\eta}\,\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = \sin {}^{\sigma}\delta^{(l)} \tag{25}$$

$$(-1)^{l+1} \frac{\pi}{\sqrt{2}} \int_0^\infty \sqrt{\xi}\, J_{-(l+\frac{1}{2})}(\sqrt{\eta}\,\xi) \cdot {}^{\sigma}B^{(l)} \cdot {}^{\sigma}R^{(l)}\, d\xi = \cos {}^{\sigma}\delta^{(l)} \lim\left[\frac{\sqrt{2}\,{}^{\sigma}R^{(l)}}{\sqrt{\xi}\, J_{l+\frac{1}{2}}(\sqrt{\eta}\,\xi)}\right]_{\xi=0}; \tag{26}$$

in deriving the last term of (26) use has further been made of the property of Bessel functions (W & W, *17.2*)

$$\lim_{z\to 0} J_n(z) J_{-n}(z) = \frac{1}{\Gamma(1+n)\,\Gamma(1-n)} = \frac{\sin n\pi}{n\pi}.$$

5.13. *General behaviour of eigenfunctions of S-states.* The wave-

* The following identities have been derived, as well as slightly more general ones, by HULTHÉN [44c, 45a].

equation (10) gives directly the curvature of the integral curve $R(\xi)$ in each point:

$$\frac{d^2R}{d\xi^2} = -(\eta + B)R \quad , \quad B = b\,w(\xi). \tag{27}$$

Starting with a solution which behaves like $a\xi$ at the origin (*5.12*) (where, for definiteness, we take $a > 0$), we are thus able to follow the whole course of the integral curve in its dependence on the potential $B(\xi)$.

5.131. *Repulsive potential.* If the potential is repulsive at all distances ($B < 0$), it is clear that no negative eigenvalue η is possible; since otherwise we would have $d^2R/d\xi^2 > 0$, i.e. the curve would always bend upwards, and consequently exhibit an exponential *in*crease at large distances. For positive values of η, the curve (fig. *5.131*) will generally * begin by bending upwards, until the point ξ_0 is reached, for which $B(\xi_0) + \eta = 0$; at ξ_0 the curve has an inflexion point and bends downwards until $R(\xi)$ itself becomes zero; and from this point it goes on oscillating in a fashion rapidly approximating the asymptotic form (21); the phase $\delta^{(0)}$ is seen to be negative.

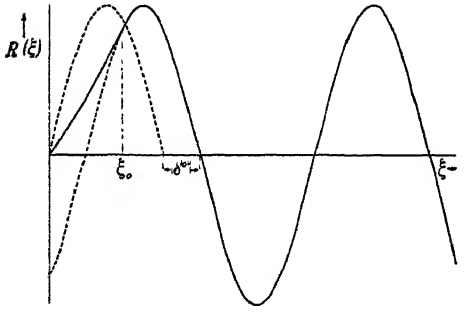


Fig. 5.131. *S*-state; repulsive potential.

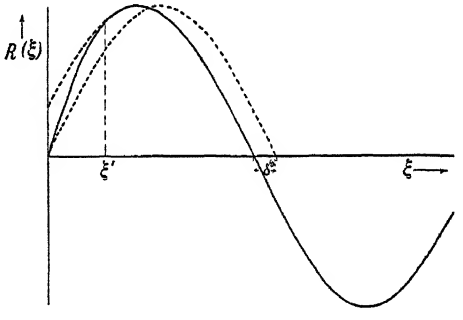


Fig. 5.132–1. *S*-state of relative motion ($\eta > 0$); attractive potential.

5.132. *Attractive potential.* If the potential is attractive at all distances ($B > 0$), the integral curve begins by bending downwards **, so that $R(\xi)$, after passing through a maximum, will begin to decrease. Everything now depends, however, on the rapidity of this variation, which in its turn is conditioned by the strength of the potential. Let ξ' be some value of ξ for which the potential becomes so small that from this point on the wave-function may be considered to be sufficiently approximated by its asymptotic form. If when reaching ξ' the value of R is still increasing, the

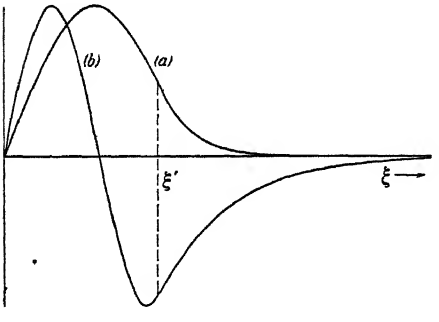
* The only exception would be, for a potential finite at the origin, the case $\eta > |b|$. But this case can readily be discussed in the same way as the others and shows the same general behaviour.

** The reader will readily recognize that the only exceptional case, viz. that of a potential finite at the origin and a negative eigenvalue with $|\eta| > b$, can be excluded.

integral curve cannot, of course, go over smoothly into an exponentially decreasing one, but must be of the periodic type (fig. 5.132–1): this will necessarily happen for sufficiently small values of b (and of η, if $\eta > 0$).

But if b is large enough, it will become possible (fig. 5.132–2, curve (a)) for the wave-function to begin decreasing before the point ξ' and to go over at this point into the exponential decrease required for the existence of a stationary state with negative eigenvalue; the latter quantity will then be determined by the slope of the integral curve at the point of junction ξ'. We thus come to the conclusion that if the potential constant b remains below a certain critical value b_1, there can be no state of binding of the deuteron. As soon as b exceeds b_1 there will be one such state; the case $b = b_1$ is a limiting case, for which the binding energy is zero.

As b increases beyond b_1, the eigenvalue corresponding to the state of binding under discussion must increase in absolute value, in order to prevent a too rapid bending of the integral curve *. At the same time, however, the possibility arises of another type of integral curve, in which the curve, after passing its first maximum, slopes down till it crosses the ξ-axis on the left of ξ', and further proceeds bending upwards on the



5.132–2 S-states of binding ($\eta < 0$); attractive potential.

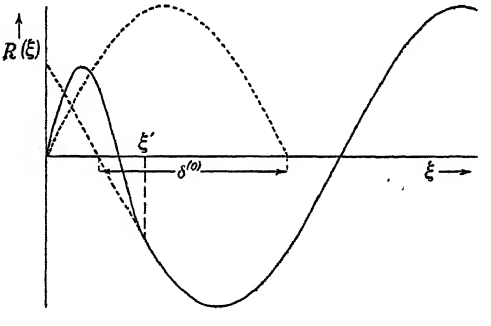


Fig. 5.132–3. S-state of relative motion ($\eta > 0$); strong attractive potential.

negative side: if by the point ξ' it has not yet reached its minimum (fig. 5.132–3), again only a periodic asymptotic behaviour, with positive eigenvalue, is possible: but there will clearly be a second critical value b_2 ($> b_1$) of b, beyond which the minimum will have been passed when ξ' is reached and an exponential decrease (on the negative side) becomes possible (fig.

* It is clear that if b tends to infinity, $|\eta|$ likewise will become infinite. From this fact we may infer the impossibility of "contact" interaction (4.31). For a "contact" potential of the form $\delta(\vec{x})$ may be approximated by potential "wells" of decreasing width $\varkappa^{-1}$ and increasing depth J, in such a way that $\lim J \cdot (\varkappa^{-1})^3 = \text{const.}$ or, by (9), $\lim_{\varkappa \to \infty} b/\varkappa = \text{const.}$ The "contact" potential, therefore, corresponds to $b \to \infty$ and does not give rise to any stationary state of finite binding energy. A more complete discussion of this question, based on the exact relativistic equations of the two-body problem, is given by KEMMER [37a].

5.132–2, curve (b)). For $b > b_2$ we shall thus have two different states of binding, distinguished by the number (0 or 1) of nodes of the eigenfunction; the stronger binding corresponds to the 0-node eigenfunction. Repeating the argument, we are finally led to the following picture: there is a succession of increasing critical values $b_0 = 0$, $b_1, b_2, \ldots$ of the potential constant b, such that if $b_n < b < b_{n+1}$, there are n S-states of binding of the deuteron, the eigenfunctions of which are characterized by an increasing number of nodes, from 0 to $n-1$, the binding energy $|\eta_k|$ $(k = 1, \ldots, n)$ decreasing accordingly. As regards the states of positive energy, it is easily seen that their phase constants $\delta^{(0)}$ (in contrast to the case of a repulsive potential) are positive.

5.133. *Attractive potential modified by very short range repulsion* (fig. 2 22). The discussion of this case proceeds on the same lines as that of the preceding ones and clearly leads to general results entirely analogous to those pertaining to a purely attractive potential (*5.132*). Only the critical values of the potential for which a new state of binding becomes possible will here be determined by the depth of the minimum of the potential function. It is also apparent that the influence of the repulsive potential at small distances will be the more important on the eigenfunction of a state of relative motion, the larger the energy of the system in this state.

5.14. *Eigenfunctions of states of higher orbital momentum.* For the discussion of the stationary states of quantum number $l \neq 0$, we conveniently introduce the *impact parameter* r_l, defined as the closest distance of approach of the two particles. On account of conservation of angular momentum, we may write, with the notation (*4.32*–11),

$$p r_l = \hbar |\vec{L}|,$$

p being the relative linear momentum at r_l, given by

$$\frac{p^2}{M} = \varepsilon - {}^{\tau}V^{(l)}(r_l);$$

r_l is therefore determined by the equation

$$\varepsilon - {}^{\tau}V^{(l)}(r_l) = \frac{\hbar^2}{M} \cdot \frac{l(l+1)}{r_l^2},$$

and the corresponding reduced parameter $\xi_l = \varkappa r_l$ by the equation

$$\eta + {}^{\tau}B^{(l)}(\xi_l) = \frac{l(l+1)}{\xi_l^2}. \tag{28}$$

Comparison with equation (10) shows that at ξ_l the integral curve has an inflexion point.

At small ξ the solution which vanishes for $\xi = 0$ is (5.12) $\sim \xi^{l+1}$; the integral curve thus starts tangentially to the ξ-axis at $\xi = 0$, then (owing to the dominant term $-l(l+1)/\xi^2$) bends upwards till $\xi = \xi_l$, after which point it begins bending downwards (fig. 5.14). If now ξ_l is so

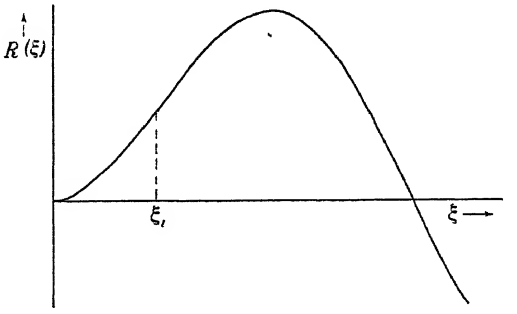


Fig. 5.14. State of orbital momentum l ($\eta > 0$).

large that the wave-function at that point already shows its asymptotic behaviour, we see that only solutions of the periodic type, with positive eigenvalues, can be adjusted at ξ_l to the upward slope of the integral curve. A state of binding, with asymptotic form $\simeq e^{-\sqrt{-\eta}\,\xi}$, can therefore only exist if the potential at ξ_l still has a sufficiently large, positive value: physically, this means, according to (28), that there must be an attractive potential sufficiently strong to overcome the repulsion due to centrifugal force. A more quantitative estimate of the required potentials may be obtained as follows: In order to get a lower limit, put the binding energy $\eta = 0$; since ξ_l is to be small, $B(\xi_l)$ is either b or $b\,\xi_l^{-1}$. In any case, the condition $\xi_l \ll 1$ yields, according to (28), $b \gg l(l+1)$. When this condition is satisfied, there are thus states of binding of orbital momenta $\leq l$ for which the potential is attractive; all states of higher l belong to the continuous spectrum of positive energies.

5.2. Stationary states of binding

5.20. After the qualitative survey of the preceding section, we go over to a more precise treatment of the eigenvalue problem of our equation (5.11–10). We shall first consider some special forms of potential allowing of an exact solution; we shall then briefly recall the general variational method often used to obtain approximate solutions. The examples treated will illustrate the general considerations of the preceding section.

5.21. *The S-states of the potential well.* If the potential is of the type (5.11–2), the eigenfunctions of the S-states of binding are of the form

$$\begin{aligned} &\sim \sin \sqrt{b - |\eta|}\,\xi \qquad (\xi \leqslant 1) \\ &\sim e^{-\sqrt{|\eta|}\,\xi} \qquad\quad (\xi > 1). \end{aligned} \tag{1}$$

The continuity conditions at $\xi = 1$ then yield

$$|\sqrt{\eta}| = -\sqrt{b-|\eta|}\,\operatorname{ctg}\sqrt{b-|\eta|}, \tag{2}$$

whence for the critical b-values

$$b_{n+1} = (n+\tfrac{1}{2})^2\pi^2. \tag{3}$$

From the normalization condition (*5.111*–11), one deduces, using also the continuity conditions at $\xi = 1$, that the cofactor of $\sin\sqrt{b-|\eta|}\,\xi$ in the normalized eigenfunction is

$$[2\sqrt{|\eta|}/(1+\sqrt{|\eta|})]^{\frac{1}{2}}. \tag{4}$$

In practice, the eigenvalue of the energy may be given by experiment, while the strength of the potential is not known. The relation (2) can then be used to derive b from $|\eta|$. In the vicinity of a critical value, $b-|\eta|$ can be expanded in powers of $\sqrt{|\eta|}$ by means of Bürmann's theorem (W & W, *7.3*). So we get for the lowest S-state (BREIT *et al.* [*36a, 39a*])

$$b = \frac{\pi^2}{4} + 2\sqrt{|\eta|} + \left(1-\frac{4}{\pi^2}\right)|\eta| + \left(\frac{32}{\pi^4}-\frac{8}{3\pi^2}\right)|\eta|^{\frac{3}{2}} + \left(\frac{32}{\pi^4}-\frac{320}{\pi^6}\right)|\eta|^2 + \ldots. \tag{5}$$

5.211. *The S-states of the exponential potential.* If the potential is of the form (*5.11*–3) $b\,e^{-2\xi}$, the substitution

$$y = \sqrt{b}\,e^{-\xi}$$

reduces the radial equation to a Bessel differential equation of order $\pm\sqrt{-\eta}$. The boundary condition at $\xi \to \infty$ then determines the eigenfunctions of bound states as Bessel functions of positive order

$$R \sim J_{+\sqrt{|\eta|}}(\sqrt{b}\,e^{-\xi}).$$

The eigenvalues are determined by the other boundary condition, $R = 0$ at $\xi = 0$:

$$J_{\sqrt{|\eta|}}(\sqrt{b}) = 0.$$

The critical b-values are accordingly the squares of the zeros of $J_0(z)$.

5.22. *The S-states of Hulthén's potential.* It has been noticed independently by HULTHÉN [*42a, b*] and by HYLLERAAS and RISBERG [41] that the potential (*5.11*–7)

$$w = \frac{e^{-\xi}}{1-e^{-\xi}}, \tag{6}$$

while representing a good approximation of the meson potential $e^{-\xi}/\xi$ (*5.11*–6), gives rise to a wave-equation which can be solved exactly. Let us make the variable substitution

$$\xi = -\log(1-x) \quad , \quad x = 1-e^{-\xi}, \tag{7}$$

and for an S-state of binding, put

$$R(\xi) = e^{-\frac{1}{2}\sqrt{|\eta|}\,\xi}\,\chi(x); \tag{8}$$

the differential equation for χ is

$$x(1-x)\frac{d^2\chi}{dx^2} - (2\sqrt{|\eta|}+1)\,x\frac{d\chi}{dx} + b\chi = 0, \tag{9}$$

with the boundary conditions

$$\chi(0) = 0;\quad \lim_{\xi\to\infty} e^{-\frac{1}{2}\sqrt{|\eta|}\,\xi}\,\chi(x) = 0 \text{ if } |\eta| > 0;\quad \chi(1) \text{ finite if } \eta = 0. \tag{10}$$

Now, define n (>0) by

$$b = n(n + 2\sqrt{|\eta|}). \tag{11}$$

If this expression (11) is substituted for b in equation (9), this equation, for an arbitrary value η_n of η, possesses a complete system of eigenfunctions $\chi_n(x)$, given by the Jacobian polynomials *

$$\begin{aligned} G_n(x) &= \sum_{\nu=1}^{n} (-1)^{\nu-1} \binom{n-1}{\nu-1} \binom{n+\nu+2\sqrt{|\eta_n|}-1}{\nu} x^\nu \\ &= -\frac{1}{n!}\,x(1-x)^{-2\sqrt{|\eta_n|}}\,\frac{d^n}{dx^n}\left[x^{n-1}(1-x)^{n+2\sqrt{|\eta_n|}}\right], \end{aligned} \tag{12}$$

n being an integer ≥ 1. If then b is considered as a given quantity, the relation (11) will determine in terms of b and n the eigenvalue η_n of η belonging to the eigenfunction χ_n:

$$-\eta_n = \frac{1}{4}\left(\frac{b}{n} - n\right)^2. \tag{13}$$

The condition that the positive square root $\sqrt{|\eta_n|}$ be taken implies, for a given b, the limitation

$$n^2 \leqslant b \tag{14}$$

on the number of eigenstates; the critical values of b are just

$$b_n = n^2. \tag{15}$$

By a simple calculation, we get for the eigenfunctions normalized according to (*5.111*–11)

$$\chi_n(x) = \sqrt{\frac{b^2-n^4}{2bn}}\,G_n(x). \tag{16}$$

The normalized eigenfunction of the ground state is therefore

$$R_1(\xi) = \sqrt{\frac{b(b^2-1)}{2}}\,e^{-\frac{b-1}{2}\xi}(1-e^{-\xi}), \tag{17}$$

with the eigenvalue

$$-\eta_1 = \tfrac{1}{4}(b-1)^2. \tag{18}$$

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* See, e.g., R. COURANT and D. HILBERT, *Methoden der mathematischen Physik I* (1924), p. 75.

5.23. *The variational method.* As is well-known, the proper solutions of equation (5.11–10) are characterized as the extremal functions of the variation problem

$$\delta \int_0^\infty R^* \left[\frac{d^2}{d\xi^2} + \eta + b\, w(\xi) - \frac{l(l+1)}{\xi^2} \right] R\, d\xi = 0 \tag{19}$$

which vanish at $\xi = 0$ and $\xi = \infty$. In this problem, the parameter b determining the strength of the potential is usually regarded as given, and the values of the parameter η are sought for which the problem has a solution. An equivalent way of stating the problem is to say that such functions R are to be determined as make the expression

$$\eta = -\frac{\int_0^\infty R^* \left[\frac{d^2}{d\xi^2} + b\, w(\xi) - \frac{l(l+1)}{\xi^2} \right] R\, d\xi}{\int_0^\infty R^* R\, d\xi} \tag{20}$$

stationary. These extremal values are then just the eigenvalues of the energy parameter η.

But it is clearly equally justified and often more convenient to regard the parameter η as given and to determine the eigenvalues of the parameter b; one has then to look for the extremal values of

$$b = -\frac{\int_0^\infty R^* \left[\frac{d^2}{d\xi^2} + \eta - \frac{l(l+1)}{\xi^2} \right] R\, d\xi}{\int_0^\infty R^* w(\xi) R\, d\xi}, \tag{21}$$

which, for negative η, may also be written, on account of the boundary conditions,

$$b = \frac{\int_0^\infty \left[\left| \frac{dR}{d\xi} \right|^2 + \left(|\eta| + \frac{l(l+1)}{\xi^2} \right) |R|^2 \right] d\xi}{\int_0^\infty |R|^2 w(\xi)\, d\xi}. \tag{22}$$

If the potential function $w(\xi)$ has always the same sign, the expression (22) is essentially positive (corresponding, physically, to an attractive potential): this property is peculiar to the form (22) of the variation problem, — the numerator of (21), e.g., is not positive definite.

The above variational principles can be used to find approximate eigensolutions and eigenvalues. A function R of suitable analytical form is chosen and made to depend on one or more parameters; the latter are then varied and the maximum or minimum problem solved. The solution obtained by inserting in the given form the extremal values of the parameters is the "best" one of the form considered and the corresponding value of η (or b) an approximation to the true eigenvalue of this quantity. By inserting the approximate solutions into the identities (*5.121*–23, 24), one gets an idea of the degree of accuracy attained.

Another method, which has never been applied to nuclear problems, but which, as suggested by Dr. FRÖHLICH, might be useful, consists in starting, instead of (19), from

$$\delta \int \left| \left[\frac{d^2}{d\xi^2} + \eta + b\, w(\xi) - \frac{l(l+1)}{\xi^2} \right] R \right|^2 d\xi = 0; \tag{19a}$$

i.e. the problem (similar to that of the "least square" method) would be to find the function R such that the sum of the absolute squares of the values taken by the left-hand side of the wave-equation (*5.11*–10) be a minimum. See also KRAMERS [38], § 51.

5.231. *The S-states of the meson potential.* As an example of the variational procedure, we shall briefly discuss the case of the meson potential (HULTHÉN [42*a*, *b*, 45*a*], HYLLERAAS and RISBERG [41]). The similarity with Hulthén's potential (*5.22*) suggests the variable substitution (7). Introducing again the function $\chi(x)$ by (8), the variational principle (21) takes the form

$$\delta b = 0 \text{ with } b = \frac{\int_0^1 \chi^* \left[(1-x) \frac{d^2}{dx^2} - (2)\sqrt{|\eta|} + 1) \frac{d}{dx} \right] \chi (1-x)^{2\sqrt{|\eta|}}\, dx}{\int_0^1 |\chi|^2 \frac{(1-x)^{2\sqrt{|\eta|}}}{\log(1-x)}\, dx}. \tag{23}$$

For the ground state, the solution $\chi = x$ corresponding to Hulthén's potential suggests the choice of the trial function

$$\chi = x - cx^2, \tag{24}$$

the best value of c being determined by the solution of the variation problem (23). One readily finds that c has to satisfy a quadratic equation, so that for every given $|\eta|$ there are two solutions for χ and b. In particular, for $|\eta| = 0$, these two b-values are the two first critical ones; they happen to be

$$b_1 \approx 1{,}68 \quad , \quad b_2 \approx 4{,}5.$$

More precisely, the relation between b and the eigenvalue $|\eta|$ can, for $|\eta| \lesssim \frac{1}{4}$, be expressed analytically by the formula

$$b = 1{,}6798 + 2{,}2655\, \sqrt{|\eta|} - 0{,}2456\, |\eta| + 0{,}1440\, |\eta|^{\frac{3}{2}} - \ldots . \tag{25}$$

Even for $|\eta| = 1$, the terms written down in (25) yield a value of b correct to 1 %. For $|\eta| \leqq \frac{1}{4}$, the contribution from the last two terms of (25) does not exceed 2 % of the b-value. More details will be found in Hulthén's papers quoted above.

It is useful to note that the wave-function $R(\xi)$ of the ground state may be represented to a fair approximation by a simple function of the form

$$R(\xi) \sim \xi e^{-w\xi},$$

with a numerical constant w. For this representation, originally proposed by WILSON [38], see also FRÖHLICH, RAMSEY and SNEDDON [46].

5.24. *The variation-iteration method.* In the case of a central (effective) potential $\mathcal{V} = -Jw(|\vec{x}|)$, the co-factors $\psi(\vec{x})$ of the spin and charge eigenfunctions $^{\sigma}(\sigma)\ ^{\tau}(\tau)$ in the total eigenfunction (*4.33*) $^3f_1$ satisfy a wave-equation of the same type (*4.33–21*) as $^3f_1$ itself. If we go over to momentum space, by

$$\chi(\vec{k}) = (2\pi)^{-\frac{3}{2}} \int \psi(\vec{x})\, e^{i\vec{k}\vec{x}}\, dv \tag{26}$$

and

$$w_k(\vec{k}) = (2\pi)^{-3} \int w(|\vec{x}|)\, e^{i\vec{k}\vec{x}}\, dv, \tag{27}$$

the wave-equation transforms into an integral equation

$$T(\vec{k})\,\chi(\vec{k}) - J \int w_k(\vec{k}-\vec{k}')\,\chi(\vec{k}')\, dv_{k'} = 0, \tag{28}$$

in which the coefficient

$$T(\vec{k}) = \frac{\hbar^2 k^2}{M} - \varepsilon \tag{29}$$

is essentially positive for a state of binding ($\varepsilon < 0$).

Upon the assumption that the (symmetrical) "kernel" $w_k(\vec{k}-\vec{k}')$ is positive definite, i.e. that

$$\int f^*(\vec{k}) \cdot w_k(\vec{k}-\vec{k}') \cdot f(\vec{k}')\, dv_k\, dv_{k'} \geqq 0 \tag{30}$$

for any $f(\vec{k})$, equation (28) can be solved by a method of successive approximations, starting from an arbitrary trial function $\chi_0(\vec{k})$ and proceeding as follows: Put, for a given ε (< 0),

$$\chi_{j+1}(\vec{k}) = \frac{1}{T(\vec{k})} \int w_k(\vec{k}-\vec{k}')\,\chi_j(\vec{k}')\, dv_{k'}$$

$$T_j = \int \chi_j^*\, T(\vec{k})\,\chi_j\, dv_k \tag{31}$$

$$W_j = \int \chi_j^*(\vec{k})\, w_k(\vec{k}-\vec{k}')\,\chi_j(\vec{k}')\, dv_k\, dv_{k'}$$

$$J_j = T_j/W_j \quad , \quad J_{j+\frac{1}{2}} = W_j/T_{j+1}.$$

It can then be shown that the sequence J_0, $J_{\frac{1}{2}}$, J_1,... never increases and

that it converges towards an eigenvalue of J; this eigenvalue is the lowest one provided the function χ_0 fulfills the condition

$$\int \chi_0^* \; T(\vec{k})\, \chi \, dv_k \neq 0, \tag{32}$$

where χ is the exact eigenfunction of the lowest state. In this case, the sequence of functions $\chi_j/\sqrt{T_j}$ converges uniformly towards the eigenfunction χ.

This method has been recently investigated by SVARTHOLM [45] and extended by him to the treatment of the many-nucleon problem, with special application to the 3- and 4-nucleon cases (*14.21*, *14.22*).

5.3. Calculation of the phases

5.30. As we have seen (*5.12*), the asymptotic behaviour of the states of relative motion is characterized by a phase constant. For the treatment of scattering processes, which depend on just the asymptotic behaviour of the wave-function, it is therefore important to obtain sufficiently accurate estimates of these phases.

5.31. *General variational method.* The variational method developed for the treatment of the states of binding (*5.23*) can, as pointed out by HULTHÉN [44*d*, 46*a*], also be adapted to the case of the states of positive energy. In our presentation of Hulthén's method, we shall follow a line of argument due to HYLLERAAS [45]. In the case of negative energies, the problem involves two mutually dependent parameters, viz. b and η, and the variation principle consists in expressing that either one of these parameters must be stationary, the other being fixed. In the case of positive energies, we have 3 dependent parameters, viz. b, η and the phase δ, the latter entering the problem through the boundary condition at infinity, which now states that the eigenfunction must have the asymptotic form (*5.12–21*) $\simeq \sin\left(\sqrt{\eta}\,\xi - l\pi/2 + {}^{\sigma}\delta^{(l)}\right)$. Again we may formulate the variation principle by requiring either b, η or δ to be stationary under the boundary condition at infinity just mentioned, the other two parameters being kept fixed.

This may be seen as follows: Let us start from (*5.23–20*) or (*5.23–21*), which means that we consider such trial functions $R(\xi;\, c_1, \ldots, c_n)$, depending on certain parameters c_i, as satisfy the equation

$$\mathcal{L}(R) \equiv \int_0^\infty R^* \left[\frac{d^2}{d\xi^2} + \eta + b\, w(\xi) - \frac{l(l+1)}{\xi^2}\right] R \, d\xi = 0; \tag{1}$$

this equation being satisfied also by the varied functions, we have, as in (*5.23–19*) *

$$\delta \mathcal{L} = 0. \tag{2}$$

* In the formulae (2), (3), (4), the letter δ occurs as a symbol of variation, and not in its usual meaning of phase constant.

In calculating $\delta\mathcal{L}$, the expression $\delta \int_0^\infty R^* \frac{d^2R}{d\xi^2}\, d\xi$ gives rise to a term

$$\left[R^* \frac{d\,\delta R}{d\xi} - \frac{dR^*}{d\xi}\,\delta R\right]_{\xi\to\infty}, \tag{3}$$

in which also the phase δ occurs. It is easily verified that this term vanishes if δ is treated as a constant. The remaining equation

$$\int_0^\infty \delta R^* [\ldots] R\, d\xi + \int_0^\infty \delta R\, [\ldots] R^*\, d\xi = 0, \tag{4}$$

with the same operator as in (1) between the brackets, shows that R is a solution of the desired wave-equation, satisfying the given boundary conditions. It can be interpreted again as expressing that either b or η must be stationary, the phase δ being kept constant. Another way of looking at the matter is to say that, starting from (1), we impose on δ the condition of being stationary for given b and η, — for this implies that δ is treated as a constant in computing (3). The latter interpretation was originally put forward by Hulthén, but both he and Hylleraas now favour the former one, which, in fact, has the advantage of reducing both cases of positive and negative energies to entirely analogous forms *.

Anyhow, the practical procedure consists in solving the system of equations

$$\mathcal{L} = 0, \qquad \frac{\partial \mathcal{L}}{\partial c_i} = 0 \qquad (i = 1, 2, \ldots, n) \tag{5}$$

for the parameters c_i and δ. In the case of a short range potential, it will be convenient to take a trial function of the general form

$$R = f(\xi; c_1, \ldots, c_n)\, \sqrt{\xi}\, J_{l+\frac{1}{2}}(\sqrt{\eta}\,\xi) + g(\xi; c_1, \ldots, c_n)\, \sqrt{\xi}\, J_{-(l+\frac{1}{2})}(\sqrt{\eta}\,\xi), \tag{6}$$

in which f and g must have the properties

$$\begin{aligned} &f \approx \text{const} \quad , \quad g \sim \xi^{2l+1} \qquad \text{for small } \xi \\ &f \simeq A \cos\delta \quad , \quad g \simeq A\,(-1)^l \sin\delta. \end{aligned} \tag{7}$$

The accuracy of the result can be checked by means of the identities (*5.121–25, 26*). The method can be improved further by imposing the first of these identities on the trial function as an accessory condition to remain fulfilled when the variation is performed. Further details and examples will be found in Hulthén's papers.

5.32. *Estimate of S-phase for nearly critical values of potential strength.* In general, the S-phases can only be derived from the explicit integration

* An important difference (HULTHÉN [46a]) is that the *stationary* parameter values, in the continuum case, do not in general correspond to a maximum or a minimum.

of the wave-equation, for instance by the method just outlined. There is, however, a case in which they can be estimated directly. Suppose the positive potential b is very near one of the critical values b_n:

$$b = b_n - \beta, \tag{8}$$

the sign of the small quantity β being arbitrary. Let $\overset{\circ}{\chi}_n$ be the radial eigenfunction of the S-state corresponding to the critical value b_n (and thus to $\eta = 0$); it is a solution of the differential equation

$$\overset{\circ}{\chi}_n'' + b_n w \overset{\circ}{\chi}_n = 0, \tag{9}$$

which we shall take to be real and normalized in such a way that

$$\overset{\circ}{\chi}_n (\infty) = 1. \tag{10}$$

If we write the eigenfunction of an S-state of positive energy η in the form

$$R(\xi) \equiv \chi(\xi) \cdot \frac{\sin(\sqrt{\eta}\,\xi + \delta)}{\sqrt{\pi}\sqrt{\eta}}, \tag{11}$$

the function χ is a solution of the differential equation

$$\chi'' + 2\sqrt{\eta}\,\operatorname{ctg}(\sqrt{\eta}\,\xi + \delta)\,\chi' + b w \chi = 0, \tag{12}$$

likewise normalized so that $\chi(\infty) = 1$ (*5.12–21*). Therefore, χ goes over into $\overset{\circ}{\chi}_n$ when $\eta \to 0$ and $\beta \to 0$.

Consider now a state of *small* positive energy η. The corresponding function $\chi(\xi)$ can then be approximated by $\overset{\circ}{\chi}_n$. Treating β and $\sqrt{\eta}$ as small quantities of the same order, we derive in the usual way, from the differential equation which must be satisfied by the next term in the expansion $\chi = \overset{\circ}{\chi}_n + \ldots$ in powers of $\sqrt{\eta}$, the "orthogonality condition"

$$2\sqrt{\eta}\,\operatorname{ctg}\delta \int_0^\infty \overset{\circ}{\chi}_n \overset{\circ}{\chi}_n' \, d\xi - \beta \int_0^\infty w \overset{\circ}{\chi}_n^2 \, d\xi = 0,$$

or, more simply,

$$\sqrt{\eta}\,\operatorname{ctg}\delta - \beta_n = 0, \tag{13}$$

since, by (10),

$$2\int_0^\infty \overset{\circ}{\chi}_n \overset{\circ}{\chi}_n' \, d\xi = [\overset{\circ}{\chi}_n(\infty)]^2 = 1;$$

in (13), we have put

$$\beta_n \equiv \beta \int_0^\infty w \overset{\circ}{\chi}_n^2 \, d\xi. \tag{14}$$

The energy η being given, condition (13) yields directly an approximate expression for the phase

$$\operatorname{tg} \delta = \frac{\sqrt{\eta}}{\beta_n}. \tag{15}$$

Now, if β is negative, there is a "last" stationary state of binding with energy η_n very small in absolute value. This energy has a more immediate physical significance than β_n, and is determined by the latter quantity. It will therefore be advantageous to express δ in terms of η_n. To find η_n, exactly the same procedure might be used as has just been applied to the derivation of δ. We would have to put for the eigenfunction

$$R_n(\xi) = \chi_n(\xi)\, e^{-\sqrt{-\eta_n}\,\xi} \tag{16}$$

and again approximate χ_n by $\overset{\circ}{\chi}_n$. A more elegant method, due to KRAMERS *, consists in regarding the eigenfunction (11) as an analytic function of the phase δ, treated as a complex quantity. For negative η, we get (since $\sqrt{\eta} = i\sqrt{|\eta|}$)

$$R \sim e^{-\sqrt{|\eta|}\,\xi} - e^{-2i\delta}\, e^{\sqrt{|\eta|}\,\xi},$$

or, using (15) and the well-known formula $e^{2i \operatorname{arctg} z} = \dfrac{1+iz}{1-iz}$,

$$R \sim e^{-\sqrt{|\eta|}\,\xi} - \frac{\beta_n + \sqrt{|\eta|}}{\beta_n - \sqrt{|\eta|}}\, e^{\sqrt{|\eta|}\,\xi};$$

the eigenvalue η_n is determined by the condition of cancelling the increasing exponential term, i.e.

$$-\eta_n = \beta_n^2. \tag{17}$$

Formula (15) for the phase constant may thus in this case be written in the form

$$\operatorname{tg} \delta = -\sqrt{\frac{\eta}{|\eta_n|}}. \tag{18}$$

If β is positive, there is no such state of binding; still, we can, for the sake of uniformity, make use of the state of positive energy $\eta_n = \beta_n^2$, and write in a similar way

$$\operatorname{tg} \delta = +\sqrt{\frac{\eta}{\eta_n}}; \tag{19}$$

to distinguish it from the actual level $-\eta_n$, the state of positive energy η_n will be called a *"virtual" level*. The phase constant is then generally

* Unpublished.

given, in terms of the energy $|\eta_n|$ of the actual or virtual level concerned, by

$$\operatorname{tg}\delta = \pm\sqrt{\frac{\eta}{|\eta_n|}}, \tag{20}$$

the $\pm$ sign being that of β, i.e. + or — according as the level is virtual or actual.

To the approximation here considered, a physical meaning can also be attached, as pointed out by HULTHÉN [42b], to the concept of virtual level: it is a state in which the mean absolute value of the potential energy $\int_0^\infty R^2 w\, d\xi$ (the wave-function being normalized in the energy scale, 5.12) is a maximum, thus corresponding to a maximum concentration of the particle density within the potential trough. This property is readily verified by inserting for R the value (11) with $\chi \approx \mathring{\chi}_n$ and $\sin^2(\sqrt{\eta}\,\xi+\delta) \approx \frac{1}{2}(1-\cos 2\delta)$; according to (14) and (15), we have thus to find the maximum of $(1-\cos 2\delta)/\operatorname{tg}\delta$, and this occurs for $\delta = 45°$, i.e. $\eta = \eta_n$. Hulthén suggested taking this extremal property as an *exact* definition of the virtual level; but we shall see presently that such a definition is not necessarily the most adequate when dealing with higher approximations.

A simple geometrical interpretation of $\operatorname{tg}\delta$ for S-states of very small energies is obtained by noticing that the tangent at the point ξ' where the eigenfunction $R(\xi)$ practically assumes its asymptotic form (5.12–21) cuts the ξ-axis at a point of abscissa

$$\xi' - \left(R\Big/\frac{dR}{d\xi}\right)_{\xi=\xi'} = \xi' - \frac{1}{\sqrt{\eta}}\operatorname{tg}(\sqrt{\eta}\,\xi' + \delta),$$

which, for small η, reduces to

$$\xi_a \equiv \varkappa a = -\operatorname{tg}\delta/\sqrt{\eta}. \tag{21}$$

According to (20), we have in first approximation,

$$\xi_a = \mp 1/\sqrt{|\eta_n|}: \tag{22}$$

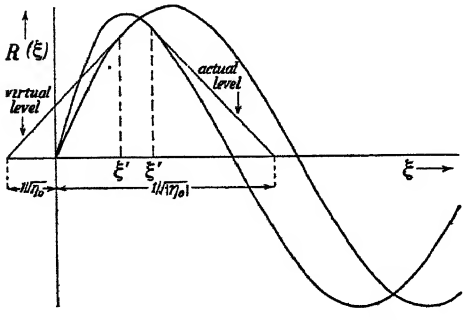

Fig. 5.32. Actual and virtual levels of small energy. The curves represent in both cases the eigenfunction of a state of small positive energy.

the point in question thus falls on the positive or negative side of the ξ-axis according as the level η_n is actual or virtual (fig. 5.32). In ordinary

units, the characteristic length a_0 corresponding to this approximation is given by

$$a_0 = \mp \frac{\hbar}{\sqrt{M|\varepsilon_n|}}. \tag{22a}$$

5.321. *Calculation of S-phases for a potential well.* In the case of a potential well (*5.11–2*; *5.21*), the calculation of the *S*-phases can be carried out explicitly. The eigenfunction of an *S*-state of positive energy is

$$\begin{aligned} &\sim \sin \sqrt{b+\eta}\,\xi \qquad (\xi < 1) \\ &\sim \sin (\sqrt{\eta}\,\xi + \delta) \qquad (\xi > 1); \end{aligned} \tag{23}$$

the continuity conditions at $\xi = 1$ yield

$$\sqrt{b+\eta} \operatorname{ctg} \sqrt{b+\eta} = \sqrt{\eta} \operatorname{ctg} (\sqrt{\eta} + \delta), \tag{24}$$

whence

$$\operatorname{ctg} \delta = \frac{\sqrt{\eta} \sin \sqrt{\eta} + \sqrt{b+\eta} \operatorname{ctg} \sqrt{b+\eta} \cos \sqrt{\eta}}{\sqrt{\eta} \cos \sqrt{\eta} - \sqrt{b+\eta} \operatorname{ctg} \sqrt{b+\eta} \sin \sqrt{\eta}}. \tag{25}$$

Suppose, now, that b is such that there is an actual level of small binding energy $|\eta^{(0)}|$. We may then, using the series for b given in (*5.21–5*), expand in powers of $\sqrt{|\eta^{(0)}|}$. For the quantity of physical interest $\sin^2\delta = (1 + \operatorname{ctg}^2\delta)^{-1}$, the result is (KITTEL and BREIT [39c])

$$\sin^2 \delta = \frac{\eta}{\eta + |\eta^{(0)}|} \left[1 + \sqrt{|\eta^{(0)}|} + G_2 |\eta^{(0)}| + G_3 |\eta^{(0)}|^{\frac{3}{2}} + \ldots\right] \tag{26}$$

with

$$\begin{aligned} G_2 &= \left(1 - \frac{4}{\pi^2}\right) - \frac{1}{4} \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} = 0{,}5947 - 0{,}25 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ G_3 &= \left(\frac{1}{3} - \frac{4}{\pi^2} + \frac{32}{\pi^4}\right) + \left(\frac{1}{\pi^2} - \frac{1}{3}\right) \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ &= 0{,}2566 - 0{,}2320 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ G_4 &= \left(\frac{1}{3} - \frac{4}{\pi^2} + \frac{48}{\pi^4} - \frac{320}{\pi^6}\right) + \left(-\frac{1}{4} + \frac{2}{\pi^2} - \frac{8}{\pi^4}\right) \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} \\ &\quad + \left(-\frac{1}{48} + \frac{1}{2\pi^2}\right) \left(\frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|}\right)^2 \\ &= 0{,}0880 - 0{,}1295 \frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|} + 0{,}0298 \left(\frac{\eta + |\eta^{(0)}|}{|\eta^{(0)}|}\right)^2, \ldots . \end{aligned} \tag{27}$$

In the case of a virtual level, $b = \pi^2/4 - \beta$, we readily find, in first approximation, $\eta^{(0)} = \beta^2/4$. If we took Hulthén's extremal property as an

exact definition of the virtual state, we should get in the next approximation (HULTHÉN [42b])

$$\eta^{(0)} = \frac{\beta^2}{4}\left[1 - \frac{\beta}{2}\left(1 - \frac{4}{\pi^2}\right)\right].$$

But in order to obtain an expansion closely parallel to the above, it is more convenient to start from a slightly different definition (KITTEL and BREIT [39c]), analogous to the relation (5.21–2) valid for an actual level:

$$\sqrt{b - \eta^{(0)}}\ \mathrm{ctg}\ \sqrt{b - \eta^{(0)}} = \sqrt{\eta^{(0)}}, \tag{28}$$

which gives instead

$$\eta^{(0)} = \frac{\beta^2}{4}\left[1 + \frac{\beta}{2}\left(1 - \frac{4}{\pi^2}\right) + \ldots\right]. \tag{29}$$

The expansion for $\sin^2\delta$ is then

$$\sin^2\delta = \frac{\eta}{\eta + \eta^{(0)}}\left[1 - \sqrt{\eta^{(0)}} + G_2\,\eta^{(0)} - G_3\,(\eta^{(0)})^{\frac{3}{2}} + \ldots\right] \tag{30}$$

with the same expressions as above for the G's.

5.3211. *Calculation of S-phases for an exponential potential* (RARITA and PRESENT [37]). Treating the problem as in 5.211, we take as eigenfunction of a state of relative motion the linear combination of Bessel functions

$$c_1 J_{i\sqrt{\eta}}(y) + c_2 J_{-i\sqrt{\eta}}(y) \qquad (y = \sqrt{b}\, e^{-\xi}).$$

The ratio of the coefficients c_1, c_2 is fixed by the boundary condition at $\xi = 0$:

$$c_1 J_{i\sqrt{\eta}}(\sqrt{b}) + c_2 J_{-i\sqrt{\eta}}(\sqrt{b}) = 0,$$

while the phase δ is defined by the asymptotic form

$$\lim_{y \to 0}\left[c_1 J_{i\sqrt{\eta}}(y) + c_2 J_{-i\sqrt{\eta}}(y)\right] \simeq \sin(\sqrt{\eta}\,\xi + \delta).$$

This means that δ is the argument of the complex quantity

$$J_{i\sqrt{\eta}}(\sqrt{b})\,\frac{e^{-i\sqrt{\eta}\log\sqrt{b}/2}}{\Gamma(1 + i\sqrt{\eta})}.$$

Hence, in the limit of very small $\sqrt{\eta}$, we readily derive for $\mathrm{tg}\,\delta$ an expression of the general form (20) or (21), with

$$\xi_a = -\frac{1}{J_0(\sqrt{b})}\left[\frac{\pi}{2} N_0(\sqrt{b}) - J_0(\sqrt{b})\left(\log\frac{\sqrt{b}}{2} + 0{,}5772\ldots\right)\right],$$

N_0 denoting the Neumann function of order zero.

5.322. *Calculation of S-phases for Hulthén's potential.* The theory of 5.32 can also be illustrated by the case of Hulthén's potential (5.11–7; 5.22). For $\overset{\circ}{\chi}_n$ we have, in virtue of (5.22–8,12),

$$\overset{\circ}{\chi}_n = \frac{(-1)^n}{n!}\, x\, \frac{d^n}{dx^n}\left[x^{n-1}(1-x)^n\right], \tag{31}$$

and therefore, by (14), (5.22–7),

$$\beta_n = \beta \int_0^1 \overset{\circ}{\chi}_n^2 \frac{dx}{x} = \beta \int_0^1 \frac{1}{n!} \frac{d^n \overset{\circ}{\chi}_n}{dx^n} x^{n-1} (1-x)^n \, dx = \frac{\beta}{2n}. \tag{32}$$

For small values of η, we then get the phase by applying formula (15).

5.323. *Calculation of S-phases for the meson potential.* In this case (*5.231*), we write the wave-function of a state of *small* positive energy in the form (11) with

$$\chi(x) = \frac{x - cx^2}{1-c}, \tag{33}$$

the variable x being again defined by (*5.22–7*). If now

$$b = 1{,}68 - \beta \qquad (|\beta| \ll 1), \tag{34}$$

the variational method outlined in *5.31* leads (HULTHÉN [42*b*]) to

$$\operatorname{tg} \delta = \frac{2{,}266 \sqrt{\eta}}{\beta} \left(1 - 0{,}125\,\beta - 3{,}885 \frac{\eta}{\beta} + \ldots\right). \tag{35}$$

Suppose that $\beta > 0$, so that we have a virtual level of low energy η_1. By putting $\operatorname{tg} \delta = 1$ and $\eta = \eta_1$ in (35), solving for β and introducing the expression obtained in (34), we get a relation, entirely analogous to (*5.231–25*), between the potential strength b and the energy η_1 of the virtual level

$$b = 1{,}6798 - 2{,}2655 \sqrt{\eta_1} + 0{,}6416\,\eta_1 - \ldots;$$

in particular, the identity (apart from the sign) of the coefficients of $\sqrt{\eta}$ in the two formulae is in accordance with the theory of *5.32*.

5.33. *Estimate of phases of states with higher orbital momentum.* In actual cases, the potential ${}^{\sigma}B^{(l)}(\xi)$ will, at any rate for $l \geq 2$, be negligible for the value of ξ given by the impact parameter ξ_l (*5.14–28*). If this is the case, we can directly use the identity (*5.121–25*) to obtain an approximate value of ${}^{\sigma}\delta^{(l)}$. In fact, we may then, in the integral, replace ${}^{\sigma}R^{(l)}(\xi)$ throughout by $\sqrt{\frac{\xi}{2}} J_{l+\frac{1}{2}}(\sqrt{\eta}\xi)$: for $\xi < \xi_l$, because both functions behave approximately as $\sim \xi^{l+1}$; for $\xi > \xi_l$, because ${}^{\sigma}B^{(l)}$ is negligible, so that ${}^{\sigma}R^{(l)} \approx \sqrt{\frac{\xi}{2}} J_{l+\frac{1}{2}}(\sqrt{\eta}\,\xi)$. We get accordingly

$$\sin {}^{\sigma}\delta^{(l)} \approx \frac{\pi}{2} \int_0^{\infty} [\sqrt{\xi}\, J_{l+\frac{1}{2}}(\sqrt{\eta})]^2 \cdot {}^{\sigma}B^{(l)} \, d\xi, \tag{36}$$

or, with the ordinary (not dimensionless) variables,

$$\sin {}^{\sigma}\delta^{(l)} \approx -\frac{\pi}{2} \cdot \frac{M}{\hbar^2} \int_0^{\infty} [J_{l+\frac{1}{2}}(kr)]^2 \cdot {}^{\sigma}V^{(l)} \, r \, dr, \tag{37}$$

on account of (*5.11–1,9*), (*5.111–13*).

Formula (36) or (37) corresponds to the so-called *Born approximation*. According to the assumption made about ${}^{\sigma}B^{(l)}(\xi)$, its validity is restricted to nuclear potentials insensible at a distance

$$\xi_l \approx \frac{\sqrt{l(l+1)}}{\sqrt{\eta}}, \quad \text{or} \quad r_l \approx \frac{\sqrt{l(l+1)}}{k}, \tag{38}$$

i.e. for sufficiently large l or sufficiently small k. The phase ${}^{\sigma}\delta^{(l)}$ given by (36) is always a small quantity; for the integrand in (36) is small for all ξ: for $\xi < \xi_l$ on account of $\sqrt{\xi} J_{l+\frac{1}{2}}(\sqrt{\eta}\,\xi)$ being small; for $\xi > \xi_l$ because ${}^{\sigma}B^{(l)}$ is then negligible.

5.331. *More accurate estimate.* A more accurate estimate can be obtained by a variant of Hulthén's method (*5.31*), proposed by PAIS [46]. It consists in taking as trial function

$$R \approx \sqrt{\frac{\xi}{2}} J_{l+\frac{1}{2}+\Delta}(\sqrt{\eta}\,\xi); \tag{39}$$

according to (*5.12*–20,21), the parameter Δ is directly related to the phase δ by

$$\delta = -\frac{\pi}{2}\Delta. \tag{40}$$

It therefore represents only a small change in the order of the Bessel function, and in conformity with the general procedure (*5.31*–5), it can be determined from the equation (*5.31*–1), in which the value (39) for R is inserted. Using a property of Bessel functions *, this gives

$$\frac{1}{2}\frac{\Delta(\Delta+2l+1)}{2\Delta+2l+1} + \int_0^\infty B R^2\, d\xi = 0. \tag{41}$$

The function (39) can be expected to be a good approximation when the influence of the nuclear forces is smaller throughout than that of the centrifugal forces. If Δ is very small, equations (40), (41) reduce to a form practically equivalent to the Born approximation (36), on account of $\sin\delta \approx \delta$.

For the meson potential $B = be^{-\xi}/\xi$, e.g., the integral in (41) becomes

$$\tfrac{1}{2} b \int_0^\infty J^2_{l+\frac{1}{2}+\Delta}(\sqrt{\eta}\,\xi)\, e^{-\xi}\, d\xi = \frac{b}{2\pi\sqrt{\eta}} Q_{l+\Delta}\left(1+\frac{1}{2\eta}\right), \tag{42}$$

$Q_n(x)$ denoting a Legendre function of the second kind **. For $x > 1$, one has (W & W, *15.31*, p. 317)

$$Q_n(x) = \frac{\sqrt{\pi}}{(2x)^{n+1}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)}\frac{1}{x^2} + \ldots\right]. \tag{43}$$

We must insert the value (42) of the integral, expanded according to (43), into equation (41) and solve for Δ. The Born approximation in this case is

$$\delta \approx \frac{b}{2\sqrt{\eta}} Q_l\left(1+\frac{1}{2\eta}\right). \tag{44}$$

* Cf. G. WATSON, *Treatise on the theory of Bessel functions* (1922), p. 403.

** Cf. G. WATSON, *l.c.*, p. 389.

CHAPTER VI

PHYSICAL PROPERTIES OF THE PROTON-NEUTRON SYSTEM

6.0. In the present Chapter, we shall discuss the available empirical material about the proton-neutron system on the assumption of a static central interaction between proton and neutron, with a view to deriving as much information as possible concerning the form of this interaction.

6.1. The ground state of the deuteron

6.11. *Energy and angular momentum.* The most precise determination of the binding energy of the deuteron in its normal state is afforded by the study of the disintegration of this nucleus by γ-rays: the threshold energy of the latter gives directly the *binding energy* (MYERS and VAN ATTA [42], WIEDENBECK and MARHOEFER [45])

$$|\varepsilon_0| = 2{,}185 \pm 0{,}006 \text{ MeV}. \tag{1}$$

The value 1 found (*1.22*) for the angular momentum of the ground state is compatible, according to the parity of the state, either with a (triplet or singlet) P-type or, for the most general static interaction, with a mixture ${}^3S + {}^3D_1$ (*4.331–22*). The former eventuality would present itself if the interaction were described by a Majorana potential with a repulsive distance dependence (*4.341*); but it is ruled out *a posteriori* by the general consistency of an interpretation of the empirical data based on the assumption that we are dealing with an *even state*. In particular, a P state would give rise to an electric quadrupole moment much larger than observed.

6.12. *Electromagnetic properties.* The next question is whether one has actually to deal with a mixture of S and D components, or with a pure S or D state, i.e. whether or not the interaction essentially involves a non-central coupling. Definite conclusions have been reached in this respect as a result of Rabi's investigations on the electromagnetic behaviour of hydrogen and deuterium molecules. His powerful magnetic resonance method of analysis of molecular beams deflected by an inhomogeneous magnetic field (RABI *et al.* [39, 40]) yields very accurate information on the energies of the stationary states of the molecules investigated, including not only the interaction with the external magnetic field, but also the mutual interaction of the constituent nuclei and electrons due to their magnetic dipole and electric quadrupole moments (*A2.1*). The discussion of the results obtained with H_2, HD and D_2 molecules led not only to a precise determination of the *magnetic moment* of the ground state of the deuteron,

but also to the disclosure of a small *electric quadrupole moment* in this state.

The latter feature is of fundamental importance, in as much as it implies the existence of a non-central interaction between proton and neutron. In fact, it is in the first place obvious that the electric quadrupole moment, defined (*A2.13*) as the expectation value of the operator

$$Q = \tfrac{1}{4}(3z^2 - r^2) = \frac{r^2}{4}(3\cos^2\vartheta - 1) = \frac{\sqrt{4\pi}}{2\sqrt{5}}\, r^2\, Y_2^0(\vartheta) \tag{2}$$

in the substate of maximum magnetic quantum number, would vanish if the ground state were a pure ^3S state; this remains true when the relativistic correction of the ^3S eigenfunction, which involves P components (*4.34–29b*), is taken into account (MØLLER and ROSENFELD [40]) *.

Let us therefore consider a general $^3S + {}^3D_1$ mixture, represented by the (unrelativistic) eigenfunction

$$\psi = \psi_S \cos\omega + \psi_D \sin\omega\,; \tag{3}$$

in this formula, ψ_S and ψ_D are the wave-functions of the ^3S and $^3D_1$ states which combine to form the ground state, both normalized to unity; $\sin^2\omega$ denotes the amount of D state admixture. For ψ_S and ψ_D we write, according to (*4.34–28*), (*4.32–17*),

$$\begin{aligned} \psi_S &= {}^1(\tau)_0 \cdot \frac{R_0(r)}{r} \cdot {}^3Z_1^{(0)\,m} = {}^1(\tau)_0 \cdot \frac{R_0(r)}{r} \frac{(-1)^m}{\sqrt{4\pi}} \cdot {}^3(\sigma)_m \\ \psi_D &= {}^1(\tau)_0 \cdot \frac{R_2(r)}{r} \cdot {}^3Z_1^{(2)\,m} \end{aligned} \tag{4}$$

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* The exact eigenfunction of a ^3S state is of the form (*4.33–18*)

$$\psi = {}^3(\varrho)_1 \cdot {}^3f_1 + {}^3(\varrho)_0 \cdot {}^3f_0 + {}^1(\varrho)_0 \cdot {}^1f_0 + {}^3(\varrho)_{-1} \cdot {}^3f_{-1}\,;$$

for the substate of magnetic quantum number $m = 1$, we have, according to (*4.34–29b*), (*4.32–14, 17*),

$${}^3f_0 = \frac{1}{\sqrt{3}}\, {}^1Z_1^{(1)1} \cdot {}^1(\tau)_0 \cdot \mathcal{D}_1\, {}^3R^{(0)} = \frac{1}{\sqrt{3}}\, {}^1(\sigma)_0 \cdot Y_1^1 \cdot {}^1(\tau)_0 \cdot \mathcal{D}_1\, {}^3R^{(0)}$$

$${}^1f_0 = \sqrt{\frac{2}{3}}\, {}^3Z_1^{(1)1} \cdot {}^1(\tau)_0 \cdot \mathcal{D}_1\, {}^3R^{(0)} = \frac{1}{\sqrt{3}} [{}^3(\sigma)_0\, Y_1^1 + {}^3(\sigma)_1\, Y_1^0]\, {}^1(\tau)_0 \cdot \mathcal{D}_1\, {}^3R^{(0)}\,.$$

The expression for the quadrupole moment, exact up to the second order in the velocities inclusively (the first order correction vanishes) involves as a factor the integral

$$\sum_{\sigma_z}' \int d\Omega\, Y_2^0 \{ |{}^3f_0|^2 + |{}^1f_0|^2 \}\,;$$

and since

$$\sum_{\sigma_z}' \{ |{}^3f_0|^2 + |{}^1f_0|^2 \} \sim 2\,|Y_1^1|^2 + |Y_1^0|^2 \sim 2 \cdot \tfrac{1}{2}\sin^2\vartheta + \cos^2\vartheta$$

is independent of ϑ, this integral vanishes.

and, by a calculation which will be developed in *6.121* below, get for the quadrupole moment $Q = \text{av } \mathcal{Q}$:

$$Q = q_1 \sin\omega\cos\omega - q_2 \sin^2\omega,$$

$$q_1 \equiv -\frac{\sqrt{2}}{10}\,\Re\int_0^\infty R_0^* R_2\, r^2\, dr \tag{5}$$

$$q_2 \equiv \frac{1}{20}\int_0^\infty |R_2|^2\, r^2\, dr.$$

Putting

$$\frac{q_1}{q_2} = \operatorname{tg}\alpha \quad , \quad \frac{2Q}{q_1} = \operatorname{tg}\xi,$$

we may transform relation (5) into

$$\cos(2\omega - \alpha) = \frac{\cos(\xi - \alpha)}{\cos\xi}. \tag{6}$$

Regarded as an equation in ω, equation (6) has two essentially distinct solutions ω_1, ω_2, connected by the relation *

$$\omega_1 + \omega_2 = \alpha.$$

Now, we may expect the quantities q_1, q_2 to be comparable in order of magnitude with $d^2 \approx 10^{-25}$ cm^2, while the value of Q deduced from the measurements of RABI and his colloborators **, viz.

$$Q = 2{,}73 \cdot 10^{-27}\ \text{cm}^2, \tag{7}$$

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* Taking into account this relation, one derives from (6)

$$\cos(\omega_1 - \omega_2) = \frac{\cos(\xi - \alpha)}{\cos\xi}$$

and further

$$\sin^2\omega_1 + \sin^2\omega_2 = 1 - \cos\alpha\cos(\omega_1 - \omega_2)$$
$$\sin^2\omega_1 - \sin^2\omega_2 = \sin\alpha\sin(\omega_1 - \omega_2),$$

so that the exact solutions of equation (6) may be expressed in the form

$$\sin^2\omega = \frac{\sin^2\alpha}{2}\left[1 - \operatorname{tg}\xi\operatorname{ctg}\alpha \pm \sqrt{1 - 2\operatorname{tg}\xi\operatorname{ctg}\alpha - \operatorname{tg}^2\xi}\,\right].$$

** The experiments yield with great accuracy the product of Q with the mean gradient of the electric field exerted on one of the nuclei of the deuterium molecule by the other nucleus and the electrons. The calculation of this factor is a delicate matter, because it requires an accurate knowledge of the electronic wave-function; it has been carried out by NORDSIECK [40]. The possible error in the estimation of this factor is the main source of uncertainty in the value of the quadrupole moment; it amounts to about 2%.

is of a much smaller order of magnitude. Accordingly, we may write, in a more or less rough approximation,

$$\omega_1 \approx \tfrac{1}{2}\xi \quad , \quad \omega_2 \approx a - \tfrac{1}{2}\xi$$

i.e.

$$\sin^2 \omega_1 \approx \left(\frac{Q}{q_1}\right)^2 \tag{8a}$$

$$\sin^2 \omega_2 \approx \frac{q_1^2 - 2\,Q\,q_2}{q_1^2 + q_2^2}. \tag{8b}$$

These solutions are of very different characters: the former represents a very small, the latter a very large modification of the ^3S eigenfunction by admixture of a $^3D_1$ wave-function.

A decision between these two possibilities can be reached by means of the empirical value of the magnetic moment. Indeed, this value (RABI *et al.* [39, 40], ARNOLD and ROBERTS [46]),

$$\mu_d = 0{,}8565 \pm 0{,}0004 \tag{9}$$

nuclear magnetons, is found to differ markedly from that of the magnetic moment of a pure ^3S state, viz.

$$\mu_d^0 = \mu_n + \mu_p = 0{,}8793 \pm 0{,}0020 \tag{10}$$

according to (*1.21*–3), (*1.22*–6): and the difference is directly interpretable in terms of an admixture of D state. In fact, for the $S + D$ mixture (3), one finds (*A2.13*–9) a magnetic moment

$$\mu_d = \mu_d^0 - \tfrac{3}{2}\,(\mu_d^0 - \tfrac{1}{2})\sin^2 \omega. \tag{11}$$

Inserting in this formula the values of μ_d and μ_d^0 given by (9) and (10), one gets

$$\sin^2 \omega = 0{,}04. \tag{12}$$

This result is compatible only with the first solution, (8a). We are therefore led to the conclusion that *the ground state of the deuteron is predominantly a ^3S state, with a small admixture of $^3D_1$ state.* It may further be observed (SCHWINGER [41]) that the fact that the D admixture cannot be *larger* than (12), together with (8a) and the numerical value (7) of the quadrupole moment, leads to an inequality

$$q_1 \gtrsim 0{,}5 \cdot 10^{-27}\ \mathrm{cm}^2, \tag{13}$$

implying a *lower* limitation of the range of nuclear interaction *.

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* Some implications of the large mixing ratio (8*b*) were discussed at an early stage (when this eventuality could not yet be excluded) by INGLIS [39*b*] and (not quite correctly) by FLÜGGE [39].

It must be emphasized, however, that the smallness of D-admixture by no means implies that the non-central coupling necessarily contributes only a relatively small correction to the binding energy of the ground state. On the contrary, several types of non-central potentials have been studied, in which the non-central terms in general play a considerable part, while nevertheless giving rise to a small D-admixture of the ^3S ground state. The discussion of such cases will, however, be postponed until we have examined the various implications of the assumption that the main part of nuclear interaction is describable by a central potential. From now on, in this and the following Part, we shall thus limit ourselves to *central interactions*; the ground state of the deuteron will accordingly be treated as a pure ^3S state.

6.121. *Calculation of the deuteron quadrupole moment.* We must still briefly indicate how the explicit calculation of the quadrupole moment can be carried out. Let us define

$$\mathcal{D}^{(12)} = (\vec{\sigma}^{(1)}\,\vec{x}_0)\,(\vec{\sigma}^{(2)}\,\vec{x}_0) - \tfrac{1}{3}\,(\vec{\sigma}^{(1)}\,\vec{\sigma}^{(2)}), \tag{14}$$

where $\vec{x}_0$ is the unit vector $\vec{x}/|\vec{x}|$. The easily established identity (RARITA and SCHWINGER [41*b*])

$$\triangle\,(r^2\,\mathcal{D}^{(12)}) = 0 \tag{15}$$

expresses that $\mathcal{D}^{(12)}$ depends on the angle variables through a linear combination of tesseral harmonics of order $l = 2$. On the other hand, $\mathcal{D}^{(12)}$ commutes with the total angular momentum and (since it is symmetrical in $\sigma^{(1)}$, $\sigma^{(2)}$) with the square of the total spin (*4.32*–8), so that it conserves the multiplicity. If, therefore, $\mathcal{D}^{(12)}$ operates on $^3Z_1^{(0)\,m}$, the resulting combination of the $Y_2^{m'}$ will just be (apart from a constant factor) $^3Z_1^{(2)\,m}$ (with the same $j = 1$ and m). The co-factor can be ascertained by explicit calculation of the coefficient of one particular $^3(\sigma)_m$ (for $m = 1$, say) in $\mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)m}$. One finds

$$^3Z_1^{(2)\,m} = -\frac{3}{2\sqrt{2}}\,\mathcal{D}^{(12)} \cdot {}^3Z_1^{(0)\,m}. \tag{16}$$

By direct calculation, one may further verify the relation

$$[\mathcal{D}^{(12)}]^2 = \tfrac{4}{9}\,(1 + P_\sigma) - \tfrac{2}{3}\,(2\,P_\sigma - 1)\,\mathcal{D}^{(12)}. \tag{17}$$

According to (2), (3) and (4), the numerical coefficients occurring in the expressions (5) for q_1 and q_2 are, respectively,

$$\frac{\sqrt{4\pi}}{\sqrt{5}}\,\Re \sum_{\sigma'} \int {}^3Z_1^{(0)\,1*} \cdot Y_2^0 \cdot {}^3Z_1^{(2)\,1}\,d\Omega$$

$$-\frac{\sqrt{4\pi}}{2\sqrt{5}} \sum_{\sigma'} \int {}^3Z_1^{(2)\,1*} \cdot Y_2^0 \cdot {}^3Z_1^{(2)\,1}\,d\Omega;$$

by (16), the first quantity becomes successively

$$-\frac{\sqrt{4\pi}}{\sqrt{5}}\cdot\frac{3}{2\sqrt{2}}\sum_{\sigma'}\int {}^3Z_1^{(0)1*}\cdot\mathcal{D}^{(12)}\cdot{}^3Z^{(0)1}\cdot Y_2^0\,d\Omega$$

$$=-\frac{\sqrt{4\pi}}{\sqrt{5}}\cdot\frac{3}{2\sqrt{2}}\cdot\frac{1}{4\pi}\cdot\frac{2}{3}\frac{\sqrt{4\pi}}{\sqrt{5}}\int (Y_2^0)^2\,d\Omega=-\frac{\sqrt{2}}{10},$$

while, by (16) and (17), the second reduces to

$$-\frac{\sqrt{4\pi}}{2\sqrt{5}}\cdot\frac{9}{8}\cdot\sum_{\sigma'}\int {}^3Z_1^{(0)1*}\cdot[\mathcal{D}^{(12)}]^2\cdot{}^3Z_1^{(0)1}\cdot Y_2^0\,d\Omega$$

$$=\frac{\sqrt{4\pi}}{2\sqrt{5}}\cdot\frac{3}{4}\sum_{\sigma'}\int {}^3Z_1^{(0)1*}\cdot\mathcal{D}^{(12)}\cdot{}^3Z_1^{(0)1}\cdot Y_2^0\,d\Omega=\frac{1}{20}.$$

6.122. *Para-ortho conversion.* For the sake of completeness, we recall here that an alternative, though less direct and much less precise method of measuring the magnetic moment of the deuteron, or rather its ratio to the proton magnetic moment, consists in studying the rate of conversion of para-hydrogen or ortho-deuterium into their respective ortho- or para-states under the influence of paramagnetic gases such as O_2 or NO (WIGNER [33*b*], KALCKAR and TELLER [35], FARKAS *et al.* [35, 38]). The ortho-para-transitions will be brought about by the inhomogeneous magnetic field of the paramagnetic molecules; the probability of their occurrence will evidently be proportional to the square of the magnetic moment of the H or D nucleus, and a comparison of the effect for H_2 and D_2 under suitably chosen conditions yields directly the ratio $|\mu_p/\mu_d|$. Owing to the approximate character of the theoretical formulae, however, the values calculated for this ratio from the empirical data are not quite definite: according to the form of theory adopted, one obtains either 3,8 or some value between 3,2 and 4; these may be compared with the figure 3,28 computed from the values of μ_d and μ_p measured in Rabi's laboratory. On the other hand, if one compares the ortho-para conversion of hydrogen and deuterium when dissolved in water or other diamagnetic liquids (FARKAS and SANDLER [39], CASIMIR [40]), it appears that besides the magnetic interaction another mechanism must be active to accelerate the deuterium conversion; a closer examination shows that the electric interaction between the dipole field of the water molecules and the deuteron quadrupole moment can account even quantitatively for the corresponding empirical result.

6.13. *Magnetic interaction of proton and neutron.* There are two *a priori* possible configurations of the deuteron which have no orbital momentum: they correspond to a triplet and a singlet S state. If the nuclear forces did not depend on the spins of the nucleons, the only energy difference between these states would be due to the different magnetic interactions of the constituent nucleons; this effect has been calculated by CASIMIR [36*a*] under the assumption that the magnetic moment of the nucleon results from a current density distribution of the form rot $\vec{\mathcal{M}}$, $\vec{\mathcal{M}}$ being the density of magnetization. There can of course

be no certainty about such an assumption as long as the theory of the anomalous magnetic moment is not better founded, but it is in any case in accordance with the charged field theory outlined in *1.21*. The interaction operator of the current distributions of proton and neutron * will then be, if $r_{PP'}$ denotes the distance of the points P, P' (*1.31–7*),

$$\mathcal{V}_{\text{magn}} = -\int \text{rot}_P \vec{\mathcal{M}}_P^{(1)} \cdot \text{rot}_{P'} \vec{\mathcal{M}}_{P'}^{(2)} \cdot \frac{dv_P\, dv_{P'}}{r_{PP'}}, \tag{18}$$

the operators $\vec{\mathcal{M}}^{(i)}$, in the configuration space $(\vec{x}^{(1)}, \vec{x}^{(2)})$ of the nucleons, having the form

$$\vec{\mathcal{M}}^{(i)}(\vec{\xi}) = \vec{\mu}^{(i)}\, \delta(\vec{\xi} - \vec{x}^{(i)}) \tag{19}$$

with $\vec{\mu}^{(i)}$ given by (*4.42–9*). We have to average this over the density distribution defined by the eigenfunctions (large components) of the ^1S and ^3S states (*4.34–28*).

For the calculation of the expectation value in such spherically symmetrical states, the operator (18) is readily found, by partial integrations, to be equivalent to

$$\begin{aligned}\mathcal{V}_{\text{magn}} &= -\frac{8\pi}{3}\int \vec{\mathcal{M}}_P^{(1)}\, \vec{\mathcal{M}}_P^{(2)}\, dv_P + \int^{\circ} (\vec{\mathcal{M}}_P^{(1)}\, \text{grad}_P)(\vec{\mathcal{M}}_{P'}^{(2)}\, \text{grad}_{P'}) \frac{dv_P\, dv_{P'}}{r_{PP'}} \\ &= -\frac{8\pi}{3}\vec{\mu}^{(1)}\vec{\mu}^{(2)}\,\delta(\vec{x}) - (\vec{\mu}^{(1)}\,\text{grad})(\vec{\mu}^{(2)}\,\text{grad})\frac{1}{r} \qquad (20) \\ &= -\frac{8\pi}{3}\vec{\mu}^{(1)}\vec{\mu}^{(2)}\,\delta(\vec{x}) - \frac{1}{r^3}\left[3\left(\vec{\mu}^{(1)}\frac{\vec{x}}{r}\right)\left(\vec{\mu}^{(2)}\frac{\vec{x}}{r}\right) - \vec{\mu}^{(1)}\vec{\mu}^{(2)}\right].\end{aligned}$$

The first term arises from a surface integral over an infinitesimal sphere with centre P, which is accordingly excluded from the domain of integration of the second term (as indicated by the sign $^{\circ}$); as it stands, the first term is correct only for the calculation of $(S \mid \ldots \mid S)$ matrix elements, but its more general expression would not offer any interest, since it does not yield any contribution to other matrix-elements, on account of the vanishing of the radial part of the wave-functions at the origin. For S states, however, it is just this first term which gives the whole expectation value, the average of the second one over all angles being zero **. The expectation

* If we assume that there is no spin-dependent interaction, there is no contribution of the exchange magnetic moment to the magnetic interaction energy (*4.43*).

** At first sight, the contribution from the second term of (20) would seem to be indeterminate, owing to the divergence of the mean value of r^{-3} in an S state. But this singularity arises only through the neglect of a relativistic correction factor $\left(1 + \frac{\varepsilon - \mathcal{V}_{\text{total}}}{M}\right)^{-2}$, $\mathcal{V}_{\text{total}}$ being the total interaction energy. The latter quantity becoming infinite (as r^{-3}) at the origin, this factor indeed secures the convergence of the mean value of r^{-3}.

value of the factor containing the isotopic variables is, of course, $\mu_n \mu_p$; as to the spin factor $\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$, or $2P_\sigma - 1$ (*4.13–26*), its average ${}^\sigma s$ is -3 for the singlet state and $+1$ for the triplet state. We finally get for the magnetic interaction energy in those states

$$ {}^\sigma\overline{V}^{(0)}_{\text{magn}} = -\mu_0^2 \mu_n \mu_p \cdot \tfrac{2}{3}\, {}^\sigma s \left| \frac{{}^\sigma R^{(0)}(r)}{r} \right|^2_{r=0} \qquad (\sigma = 1,3). \tag{21} $$

If we suppose the nuclear potential to be the same for both states, we should take ${}^1R^{(0)} = {}^3R^{(0)}$. Since $\mu_n < 0$, $\mu_p > 0$, this would mean, however, that the singlet state would be lower than the triplet state, — a conclusion in contradiction with the spectroscopic evidence indicating that the ground state has angular momentum 1 (*6.11*). This is a strong argument pointing to a significant *spin dependence of the proton-neutron potential*, large enough to depress the ${}^3S$ below the ${}^1S$ state. The order of magnitude of singlet-triplet separation due to magnetic interaction is easily estimated from (21). One may write in general *

$$ \left| \frac{R^{(0)}(r)}{r} \right|^2_{r=0} = \frac{2\varkappa^2}{|a_0|}\, \alpha, \tag{22} $$

a_0 being the characteristic length (*5.32–22a*) associated with the ground state and α a numerical factor of the order of 2 or 3. The level separation is therefore, by (21), (22),

$$ \begin{aligned} {}^3\overline{V}^{(0)}_{\text{magn}} - {}^1\overline{V}^{(0)}_{\text{magn}} &\approx 4 \cdot \tfrac{2}{3}\, \mu_0^2 |\mu_n| \mu_p \frac{2\sqrt{M|\varepsilon_0|}\,\varkappa^2}{\hbar}\, \alpha \\ &\approx \frac{4}{3}\frac{\hbar}{e^2}\left(\frac{m}{M}\right) |\mu_n| \mu_p \sqrt{m|\varepsilon_0|}\, \alpha' \\ &\approx 0{,}013\, \alpha'\ \text{MeV}, \end{aligned} \tag{23} $$

if the values (*1.21–3*), (*1.22–6*) and (1) are inserted, and $\varkappa^{-1}$ is taken to be of the order of d (we have written $\varkappa^2 a = d^{-2}\alpha'$; α' is again of the

* Formula (22) is easily obtained, in a rough approximation corresponding to the limiting case of a force of zero range, from the normalization and continuity conditions for a potential well of vanishing width. For a well of finite width, we get, according to (*5.21–1, 4, 5*)

$$ \left[\frac{R(\xi)}{\xi}\right]^2_{\xi=0} = \frac{2\sqrt{|\eta|}\,(b-|\eta|)}{1+\sqrt{|\eta|}} \approx 2\sqrt{|\eta|}\,\frac{\pi^2}{4}(1-0{,}19\sqrt{|\eta|}), $$

which, by (*5.32–22*), is of the form (22), with $\alpha \approx 2{,}2$ (taking $|\varepsilon_0| \approx 2{,}18$ MeV and $\varkappa^{-1} \approx d$). For Hulthén's potential, formulae (*5.22–17, 18*) yield

$$ [R(\xi)/\xi]^2_{\xi=0} = b(b+1)\sqrt{|\eta|} = 2\sqrt{|\eta|}\,(1+3\sqrt{|\eta|}+2|\eta|); $$

taking here for $\varkappa$ the value corresponding to a meson mass $M_m \approx 240\, m$ (*1.32–12*; *1.33–13*), i.e. $\varkappa \approx 1{,}75\, d^{-1}$, we get $\alpha \approx 2{,}4$.

order of 2 to 6). This is a quite small interval compared with the separation due to the spin dependent nuclear interaction: in fact, as we shall see, this interaction is so large that there is no ^1S state of binding of the deuteron.

6.2. Scattering of slow neutrons by protons

6.20. The evidence for the existence of a strong spin dependence of the proton-neutron interaction, discussed in the preceding section (*6.13*), is corroborated by the study of the scattering of slow neutrons by protons. In this section, we shall first develop the theory of the scattering process (in a form valid also for larger neutron velocities); we shall then summarize the revelant empirical data and discuss their implications as regards the nuclear potential.

6.21. *Theory of proton-neutron scattering.* In order to describe the process of proton-neutron scattering, we look for a solution of the two-body problem, with given total energy, asymptotically representing an incident plane wave Ψ_{inc} and a scattered spherical wave Ψ_{scatt}:

$$\Psi \simeq \Psi_{\text{inc}} + \Psi_{\text{scatt}}. \tag{1}$$

Such a solution will be a linear combination of the eigenstates of the given (positive) energy η, of the form (*4.33*–18), (*4.34*–28). To the first order in the nucleon velocities, the differential cross-section for the scattering of a given incident wave will therefore be

$$d\mathcal{S} = \sum_{\varrho_3',\,\sigma_z'} |\Psi_{\text{scatt}}|^2 r^2\, d\Omega = \sum_{\varrho_3',\,\sigma_z'} \{|\overset{(0)}{\Psi}|^2 + (\overset{(0)}{\Psi}\,\overset{(1)}{\Psi^*} + \overset{(0)}{\Psi^*}\,\overset{(1)}{\Psi})\}_{\text{scatt}}\, r^2\, d\Omega,$$

the incident wave being normalized to unity. From (*4.33*–18) it appears that the summation over ϱ_3' will in any case make the first order contribution vanish, so that we have only to compute the large component $\overset{(0)}{\Psi}_{\text{scatt}}$; in the following calculations, we shall accordingly omit the index (0) and the factor $^3(\varrho)_1$ in Ψ, and the summation over ϱ_3' (which just gives 1) in $d\mathcal{S}$, so that we simply write

$$d\mathcal{S} = \sum_{\sigma_z'} |\Psi_{\text{scatt}}|^2 r^2\, d\Omega. \tag{2}$$

The incident wave representing a proton and a neutron moving with relative momentum p in the z-direction has the asymptotic form

$$^{\sigma}\Psi_{\text{inc}} \simeq \frac{1}{\sqrt{2}} \{u_+(1)\, u_-(2)\, e^{ikz} \mp u_-(1)\, u_+(2)\, e^{-ikz}\}, \tag{3}$$

u_+, u_- being the eigenfunctions of τ_3, and the double sign being — or + according as the spins of the two particles are parallel ($\sigma = 3$, triplet system) or antiparallel ($\sigma = 1$, singlet system); k is $p/\hbar$ as given by

(*5.111–13*). With the help of the well-known expansion

$$e^{ikr} = \sum_{l=0}^{\infty} (2l+1)\, i^l P_l(\cos\vartheta) \cdot \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr)$$
$$\simeq \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\; Y_l^0 \cdot \frac{e^{ikr} - (-1)^l e^{-ikr}}{2ikr} \tag{4}$$

(the normalized spherical harmonic $Y_l^0 = \sqrt{(2l+1)/4\pi}\, P_l(\cos\vartheta)$, P_l being the corresponding Legendre polynomial; see *N.14*), we may write (3) in the form

$${}^{\sigma}\Psi_{\text{inc}} \simeq \sum_{l=0}^{\infty} \sqrt{4\pi(2l+1)}\; Y_l^0 \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr} - (-1)^l e^{-ikr}}{2ikr}, \tag{5}$$

where

$${}^{\tau}(\tau)_0 = \frac{1}{\sqrt{2}} \{u_+(1)\, u_-(2) \mp (-1)^l u_-(1)\, u_+(2)\} \tag{6}$$

is just the eigenfunction defined by (*4.331–23*). For the incident wave we may thus take as the most general combination

$$\Psi_{\text{inc}} \simeq \sum_{\sigma=1,3} \sum_{m_s} {}^{\sigma}J_{m_s} \cdot {}^{\tau}(\sigma)_{m_s} \cdot {}^{\sigma}\Psi_{\text{inc}} \tag{7}$$

with arbitrary amplitudes ${}^{\sigma}J_{m_s}$ (for $\sigma = 1$, $m_s = 0$; for $\sigma = 3$, $m_s = -1$, 0, 1).

The total wave is, according to (*4.34–28*) and (*5.12–21*), of the general asymptotic form *

$$\Psi \simeq \sum' {}^{\sigma}\mathfrak{A}_j^{(l)m} \cdot {}^{\sigma}Z_j^{(l)m} \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr + i^{\sigma}\delta^{(l)}} - (-1)^l e^{-ikr - i^{\sigma}\delta^{(l)}}}{2ir \cdot i^l}. \tag{8}$$

The condition that $\Psi - \Psi_{\text{inc}}$ represents only an outgoing spherical wave readily yields

$$\Psi_{\text{scatt}} \simeq \sum_{\sigma, m_s, l} {}^{\sigma}J_{m_s} \cdot \sqrt{2l+1} \cdot {}^{\sigma}(\sigma)_{m_s} \cdot Y_l^0 \cdot {}^{\tau}(\tau)_0 \cdot \frac{e^{ikr}}{ikr} \cdot \sqrt{\pi}\,(e^{2i^{\sigma}\delta^{(l)}} - 1). \tag{9}$$

According to (6), this may be written as

$$\Psi_{\text{scatt}} \simeq \frac{1}{\sqrt{2}}\, u_+(1)\, u_-(2) \sum_{\sigma, m_s, l} \ldots + \frac{1}{\sqrt{2}}\, u_-(1)\, u_+(2) \sum_{\sigma, m_s, l} (\mp)(-1)^l \ldots$$

Now, in a scattering experiment, the identity of the particles is fixed by the production and detection arrangements, so that e.g. particle (1) is in the neutron state, particle (2) in the proton state. We then have

* The normalization factors occurring in (*5.12–21*) have been included in the coefficients $\mathfrak{A}$.

$u_+(1) = u_-(2) = 1$, $u_-(1) = u_+(2) = 0$ and only the first sum in the expression of Ψ_{scatt} subsists. If we further assume the incident wave to be unpolarized *, all $|{}^{\sigma}J_{m_s}|^2$ are equal; normalizing to unity, we get from (3), (7) for their absolute value $\frac{1}{2}|{}^{\sigma}J_{m_s}|^2 = \frac{1}{4}$ and the differential cross-section (2) becomes **

$$d\mathcal{S} = \frac{\pi}{k^2}\, d\Omega \sum_{l,l'} \sqrt{(2l+1)(2l'+1)}\; Y_l^0\, Y_{l'}^0 \sum_{\sigma} \sigma \sin {}^{\sigma}\delta^{(l)} \sin {}^{\sigma}\delta^{(l')} \cos\left[{}^{\sigma}\delta^{(l)} - {}^{\sigma}\delta^{(l')}\right]. \quad (10)$$

For the total cross-section we get, on account of the orthogonality of the harmonics

$$\mathcal{S} = \frac{\pi}{k^2} \sum_{l} (2l+1) \sum_{\sigma} \sigma \sin^2 {}^{\sigma}\delta^{(l)}. \quad (11)$$

Each state of the given energy contributes to the total cross-section a term proportional to the square of the sine of its phase and to its "weight" $(2l+1)\,\sigma$.

The differential cross-section, giving the angular distribution of the scattered particles, consists of an expansion in terms of the successive powers of $\cos\vartheta$. Since the phases decrease with increasing l (*5.12*), the main contribution to the scattering will in any case be that of the S-waves, yielding a spherically symmetrical distribution (in the barycentric system of reference). The P-waves will mainly contribute, by interference with the S-waves, a term in $\cos\vartheta$ in the angular distribution, and so on. It is important to observe, however, that the coefficient of any power of $\cos\vartheta$ in the differential cross-section strictly speaking depends on *all* waves of different orbital momenta.

The total cross-section, on the other hand, is less sensitive to the influence of the waves of higher orbital momenta, as formula (11) shows. In the case, treated in (*5.32*), that there exists an actual or virtual S-level of *sufficiently small energy* $|\eta^{(0)}|$, we get from (*5.32*–20), (11), (*5.11*–9) and (*5.111*–13), for the corresponding contribution of the S-wave to the *scattering cross-section for not too fast neutrons*

$$\mathcal{S}^{(0)} = \frac{\pi}{k^2} \cdot \frac{\eta}{\eta + |\eta^{(0)}|} = \frac{\pi\hbar^2}{M} \frac{1}{\varepsilon + |\varepsilon^{(0)}|}, \quad (12)$$

or 3 times this quantity if the level belongs to the triplet system. In the limit of *zero energy of the scattered particles*, the scattering cross-section becomes, by (*5.32*–21) and (*5.111*–13)

$$\mathcal{S}^{(0)} = \pi a^2, \quad (13)$$

where a has the geometrical meaning explained in *5.32*; it also appears

* For the scattering of polarized neutrons, see *6.35*.

** Of course, the same result would have been obtained from the second sum of Ψ_{scatt}, corresponding to the other possible numeration of proton and neutron.

here as the radius of the "sphere of action" of a proton with respect to a very slow neutron. A more or less rough approximation to a is the quantity a_0 defined by (*5.32–22a*); for the deuteron ground state, one has

$$ {}^3a_0 = \frac{\hbar}{\sqrt{M|\varepsilon_0|}} = 0{,}44 \cdot 10^{-12}\ \text{cm}. \tag{14} $$

The binding energy of the deuteron is so large, however, that the formula (12) or the approximation $a \approx a_0$ in (13) are insufficient as regards the ${}^3S$ contribution to the scattering. The necessary correction cannot be given independently of the form of the potential; we shall now calculate it in the case of the well potential and we shall see that it is quite considerable.

6.211. *Case of the well potential.* According to the explicit calculation of the S-phase, carried out in *5.321*, we see that the right-hand side of formula (12) must be multiplied by a correction factor (*5.321–26, 30*)

$$ f = 1 \pm \sqrt{|\eta^{(0)}|} + G_2 |\eta^{(0)}| \pm G_3 |\eta^{(0)}|^2 + \ldots, \tag{15} $$

the $\pm$ sign corresponding to an actual or virtual level, respectively, and the G's being given in terms of $\varepsilon/|\varepsilon^{(0)}|$ by (*5.321–27*). If the width $\varkappa^{-1}$ of the well is denoted by D, the parameter $\pm\sqrt{|\eta^{(0)}|}$ is, by (*5.32–22*), just D/a_0 (with its sign); we may thus re-write

$$ f = 1 + \frac{D}{a_0} + G_2 \left(\frac{D}{a_0}\right)^2 + G_3 \left(\frac{D}{a_0}\right)^3 + \ldots. \tag{16} $$

Since the range D does not occur in (12), but only in the co-factor f, the expression given by (12) is often called the *zero range cross-section*, and f the *correction for finite range*.

Likewise, one has

$$ a = a_0 f_0^{\frac{1}{2}}, \tag{17} $$

where f_0 represents the series (16) for $\varepsilon = 0$; the coefficients G_i are then

$$ G_2 = 0{,}345 \quad , \quad G_3 = 0{,}025 \quad , \quad G_4 = -0{,}012 \qquad (\varepsilon = 0). \tag{18} $$

For the ${}^3S$ scattering, the general expressions of the G's may be written, on account of (*6.11–1*),

$$ \begin{aligned} G_2 &= 0{,}345 - 0{,}057\ \mathrm{E} \\ G_3 &= 0{,}025 - 0{,}053\ \mathrm{E} \\ G_4 &= -0{,}012 - 0{,}016\ \mathrm{E} + 0{,}002\ \mathrm{E}^2 \end{aligned} \tag{19} $$

(E expressed in MeV);

in these formulae, $\mathrm{E}\ (= 2\varepsilon)$ denotes the kinetic energy of the neutron with respect to the proton at rest (*6.212*). The range D being of the order d, the ratio $D/{}^3a_0$ is, by (14), of the order 0,64, which shows the importance of the finite range correction in this case.

6.212. *Transformation to laboratory system.* The preceding treatment of the scattering process utilizes the barycentric system of reference, in

which the behaviours of proton and neutron are entirely equivalent. For practical purposes, however, we must go over to a "laboratory" system of reference, in which the bulk of the scatterer is at rest. When we have to do with the scattering of neutrons of large kinetic energy (i.e. large compared to the binding energy of protons in substances like paraffin, $\approx$ 0,4 eV), the protons may be considered as free and we may take as laboratory system that in which they are at rest. The above theory is then immediately applicable and the transition from the barycentric to the laboratory system, as appears from the self-explanatory figure *6.212*, is effected in the following way *:

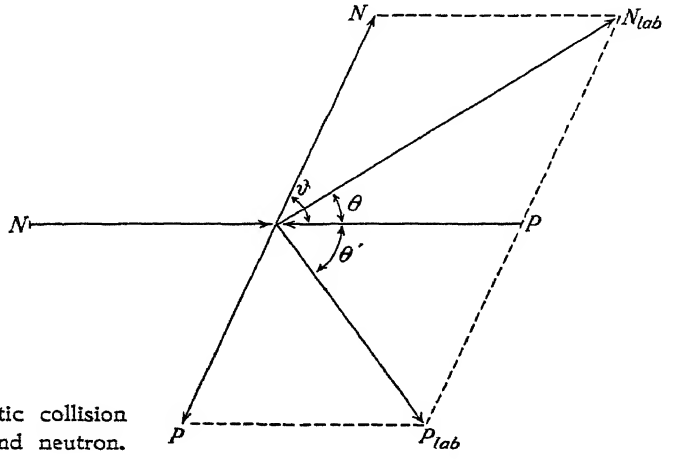

Fig. 6.212. Elastic collision between proton and neutron.

		Laboratory		Barycentric
	Energy of scattered particle	E	$=$	2ε
	Deflection angle	Θ	$=$	$\frac{1}{2}\vartheta$
	Azimuth	Φ	$=$	φ
	Element of solid angle	$4\cos\Theta\, d\Omega_{\text{lab}}$	$=$	$d\Omega_{\text{bar}}$
If recoil proton is observed	Deflection angle of recoil particle of scatterer	Θ'	$=$	$\frac{\pi}{2}-\frac{\vartheta}{2}$
	Element of solid angle for recoil particle	$4\cos\Theta' d\Omega'_{\text{lab}}$	$=$	$d\Omega_{\text{bar}}$.

(20)

The relation between the elements of solid angle expresses the fact that an angular distribution with spherical symmetry in the barycentric system corresponds to a "cos Θ -law" in the laboratory system.

In the case of very slow neutrons, however, the binding of the protons and their thermal motion becomes of considerable influence on the scattering process. This effect was first pointed out by FERMI [36] ** and further discussed by BETHE [37] (§ 59C, p. 122—127), ARLEY [38], and SACHS

* It has been assumed that particle (1) is the neutron, particle (2) the proton.

** A somewhat more accurate method than FERMI's has been developed by BREIT [47a] and BREIT and ZILSEL [47b].

and TELLER [41] *. We cannot enter into this question here, we shall only indicate, in the next sub-section, the principle of the method by which it can be treated. Empirical results obtained with very slow neutrons should thus always be corrected for the binding effect ** before being compared with the free proton formulae ***.

6.213. *Scattering of slow neutrons by bound protons.* In order to account for the binding of the proton, one has to add to the Hamiltonian of the proton-neutron system a term of potential energy depending only on the proton coordinates. At first sight, this would seem greatly to complicate the solution of the corresponding wave-equation. But an ingenious argument, due to FERMI [36], leads to a very simple treatment of the scattering problem in this case also. This argument is based on the fact that, on the one hand, the neutron wave lengths involved and, on the other hand, the distances over which the binding potential changes appreciably are both much larger than the range of the nuclear potential. The asymptotic behaviour of the wave-function † $\Psi(\vec{x}^{(n)}, \vec{x}^{(p)})$, responsible for the scattering, will thus be the same as that of a function $\Phi(\vec{x}^{(n)}, \vec{x}^{(p)})$, smoothed out by taking the mean value of Ψ with respect to the neutron coordinates over a sphere of radius intermediate between neutron wave-length and range of nuclear force. A wave-equation can then be set up for Φ, in which the influence of the nuclear potential appears only as a small perturbation.

In order to avoid unessential complications, we shall first carry out this procedure under the simplifying assumption that the nuclear potential is spin independent; we have then to do with only one scattering radius a for neutrons of zero velocity (*6.21*), the same in triplet and singlet configurations. Let us then consider some length r_0 much smaller than the neutron wave-length, but at the same time large compared both with the range $\varkappa^{-1}$ of the nuclear force and with the scattering radius $|a|$. Let Σ be the volume of a sphere of radius r_0, and

$$\Phi(\vec{x}^{(n)}, \vec{x}^{(p)}) = \frac{1}{\Sigma}\int_{\Sigma} \Psi(\vec{x}^{(n)} + \vec{x}', \vec{x}^{(p)})\, dv'. \tag{21}$$

Since Φ differs appreciably from Ψ only for such values of $\vec{x}^{(n)}, \vec{x}^{(p)}$ that the distance $r \equiv |\vec{x}^{(n)} - \vec{x}^{(p)}|$ is smaller than r_0, we may put

$$\Psi = \Phi\, \chi(r), \tag{22}$$

---

* The main formulae are not written down correctly in this last paper; their correct expression is given by GIBERT and ROSSEL [46].

** ARLEY's paper [38] contains numerical evaluations useful for the reduction of experimental data.

*** Conversely, an interesting attempt has recently been made by ROSSEL [47] to develop the study of the scattering of slow neutrons by chemical compounds in definite states of aggregation into a method of analysis of molecular and intermolecular forces.

† For convenience, we here treat proton and neutron as different kinds of particles.

the function $\chi(r)$, which expresses so to speak the distortion of the wave-function due to the nuclear potential, tending asymptotically to unity; we have further neglected the angular dependence of χ, since we know that for slow neutrons we have mainly to do with S-wave scattering. Calling $\mathcal{H}'$ the total Hamiltonian without the nuclear potential $\mathcal{V}$, we may write the wave-equation for Φ as

$$\mathcal{H}'\,\Phi + \frac{1}{\Sigma}\int_{\Sigma}[\mathcal{V}\,\Psi]_{\vec{x}^{(n)}+\vec{x}'}\,dv' = \mathsf{E}\Phi,$$

or approximately, by (22),

$$\mathcal{H}'\,\Phi + \mathcal{U}\,\Phi \approx \mathsf{E}\Phi, \tag{23}$$

with

$$\mathcal{U} = \frac{1}{\Sigma}\int_{\Sigma}[\mathcal{V}\chi]_{\vec{x}^{(n)}+\vec{x}'}\,dv'. \tag{24}$$

Combining (23) with the equation $\mathcal{H}'\,\Psi + \mathcal{V}\,\Psi = \mathsf{E}\,\Psi$ for Ψ and with (22), we get for χ, to the same approximation,

$$\left[\frac{1}{M}p^2 + \mathcal{V} - \mathcal{U}\right]\chi = 0,$$

p being the relative momentum operator. Now, we may solve this equation by disregarding the last term, since, with the solution so obtained, this term turns out to be negligible in comparison with the first. We thus have

$$\chi = \frac{R}{r}, \qquad R'' - \frac{M}{\hbar^2}\mathcal{V}R = 0, \tag{25}$$

i.e. just the radial equation for an S-wave of zero energy. Therefore, remembering the geometrical meaning of the quantity a introduced in (5.32–21) and normalizing so that $\chi(\infty) = 1$,

$$R(0) = 0, \quad R(r) \simeq -a + r. \tag{26}$$

We may now evaluate $\mathcal{U}$ in the following way; for $r > r_0$, $\mathcal{U} \approx 0$; for $r < r_0$, we approximate it by putting $r = 0$, so that

$$\text{[by (25)]} \qquad \mathcal{U} \approx \frac{4\pi}{\Sigma}\int_0^{r_0}\mathcal{V}(r')\,\chi(r')\,r'^2\,dr' = \frac{4\pi}{\Sigma}\cdot\frac{\hbar^2}{M}\int_0^{r_0}R''\,r\,dr$$

$$\text{[by (26)]} \qquad = \frac{4\pi}{\Sigma}\cdot\frac{\hbar^2}{M}\,[R'\,r - R]_0^{r_0} = \frac{4\pi\hbar^2 a}{M}\cdot\frac{1}{\Sigma}.$$

Denoting by $\delta_{r_0}(r)$ the function

$$\delta_{r_0}(r) = \frac{1}{\Sigma} \text{ for } r < r_0$$
$$= 0 \quad \text{for } r > r_0, \tag{27}$$

so that $\int_\infty \delta_{r_0}(r)\,dv = 1$, we may thus write

$$\mathcal{U} \approx \frac{4\pi\hbar^2 a}{M}\,\delta_{r_0}(r); \tag{28}$$

this is the potential which in the equation (23) for Φ represents the influence of the nuclear forces. It is easily seen that in comparison with the other terms of the equation, it can be regarded as a small perturbation throughout. We may therefore apply to the scattering problem the Born method of approximation. Consider the process by which a neutron of wave-vector $\vec{k}_0$ ($\equiv \vec{p}_0/\hbar$) is scattered with wave-vector $\vec{k}_1$, the proton going over from the state 0 of binding energy $-\mathsf{E}_0$, represented by the eigenfunction $\varphi_0(\vec{x}^{(p)})$, to the state 1 of energy $-\mathsf{E}_1$ and eigenfunction $\varphi_1(\vec{x}^{(p)})$; the partial cross-section for this process is given by

$$d\mathcal{S}(\vec{k}_0; 0 \to 1) = \frac{k_1}{k_0}\,|f(\vec{k}_0; 0 \to 1)|^2\,d\Omega^*$$

$$f(\vec{k}_0; 0 \to 1) = \frac{M^*}{2\pi\hbar^2}\int e^{i(\vec{k}_0 - \vec{k}_1)\vec{x}^{(n)}}\,\varphi_1^*(\vec{x}^{(p)})\,\mathcal{U}(|\vec{x}^{(n)} - \vec{x}^{(p)}|)\,\varphi_0(\vec{x}^{(p)})\,dv^{(n)}\,dv^{(p)} \tag{29}$$

$$\frac{\hbar^2}{2M^*}(k_0^2 - k_1^2) = \mathsf{E}_0 - \mathsf{E}_1.$$

In this formula, $d\Omega^*$ refers to the system of reference in which the centre of gravity of the total system (neutron + proton + system to which proton is bound) is at rest, and M^* denotes the reduced mass of the neutron in this system of reference, which may be taken as "laboratory system". Inserting the value (28) for $\mathcal{U}$ and replacing $\delta_{r_0}(r)$ by the δ-function (which corresponds to $r_0 \to 0$), we get

$$f(\vec{k}_0; 0 \to 1) = 2a\,\frac{M^*}{M}\int e^{i(\vec{k}_0 - \vec{k}_1)\vec{x}}\,\varphi_1^*(x)\,\varphi_0(x)\,dv. \tag{30}$$

This theory is easily generalized to the case of the scattering of a slow neutron by a bound nucleus of mass A: we have then simply to replace, in formulae (28) and (30), the reduced mass $\frac{1}{2}M$ by $\frac{A}{A+1}M$; the general meaning of the scattering radius a is that the expression $\mathcal{S}_{\text{free}} = 4\pi a^2$ represents the total cross-section for scattering of slow neutrons by *free* nuclei of mass A. In particular, in the limiting case of infinitely tight binding

of the scattering nuclei, we have only elastic scattering, and $M^* = M$; the differential cross-section (29), (30) reduces to the form

$$d\mathcal{S}_\infty = \left(\frac{A+1}{A}\right)^2 a^2 \, d\Omega^*: \tag{29a}$$

the scattering is spherically symmetrical in the *laboratory* system and the total scattering cross-section becomes

$$\mathcal{S}_\infty = \left(\frac{A+1}{A}\right)^2 \mathcal{S}_{\text{free}}. \tag{29b}$$

For tightly bound protons, for instance, it is 4 times that for free protons; for tightly bound deuterons, the corresponding factor is 2,25.

We have still to indicate how the influence of spin dependent nuclear forces is taken into account. In the case of scattering by protons, this means that the scattering radius has different values ${}^\sigma a$ for the singlet and triplet configurations. For a given spin transition, we have then to replace a in (30) by the corresponding matrix-element of the operator

$$a = {}^3a \cdot \tfrac{1}{2}(1 + P_\sigma) + {}^1a \cdot \tfrac{1}{2}(1 - P_\sigma),$$

P_σ being the exchange operator for the spin variables; by (*4.13–26*), this may be written

$$a = \tfrac{1}{4}(3 \cdot {}^3a + {}^1a) + \tfrac{1}{4}({}^3a - {}^1a)\,\vec{\sigma}^{(p)}\,\vec{\sigma}^{(n)}. \tag{31}$$

This formula is readily extended to the case that one of the interacting particles has an intrinsic angular momentum $\vec{S}$, with quantum number S different from $\frac{1}{2}$; one finds

$$a = \frac{S+1}{2S+1}\,{}^{(2S+2)}a + \frac{S}{2S+1}\,{}^{(2S)}a + ({}^{(2S+2)}a - {}^{(2S)}a)\,\frac{\vec{\sigma}\,\vec{S}}{2S+1}. \tag{31a}$$

The mean value of a^2 for all possible spin states of the colliding particles is

$$\overline{a^2} = \frac{S+1}{2S+1} \cdot {}^{(2S+2)}a^2 + \frac{S}{2S+1} \cdot {}^{(2S)}a^2. \tag{31b}$$

For scattering of slow neutrons by bound deuterons, e.g., we have to take

$$a = \tfrac{1}{3}(2 \cdot {}^4a + {}^2a) + \tfrac{1}{3}({}^4a - {}^2a)\,\vec{\sigma}^{(n)}\,\vec{S} \tag{31c}$$

and, instead of (28),

$$\mathcal{U} \approx \tfrac{3}{2} \cdot \frac{2\pi\hbar^2 a}{M}\,\delta(\vec{x}^{(n)} - \vec{x}^{(d)}). \tag{31d}$$

6.22. *Experiments on the scattering of slow neutrons by protons.* Numerous determinations of the scattering cross-section of protons for neutrons of various energies have been carried out. The arrangement of

all such experiments is quite straightforward: between the neutron source and the detector, a sample of the material to be tested is introduced and the resulting decrease of the neutron intensity is measured; from this, the scattering cross-section is easily deduced. In our case, the scatterer will consist of some hydrogenous substance, such as paraffin or water and the

6.22–1. *Scattering of thermal neutrons by protons*

Reference	Scatterer	$\mathcal{S}'$ 10^{-24} cm²	$\mathcal{S}$ 10^{-24} cm²
GOLDHABER and BRIGGS [37]	paraffin	47,5 ± 5	17,7 ± 2
FRISCH *et al.* [37]	paraffin	48 ± 3	18 ± 1,5
POWERS *et al.* [38]	≈ $C_{16}H_{34}$	42,5 ± 0,9	15,8 ± 0,4
CAROLL and DUNNING [38]	C_4H_{10}	50,0	18,6
	≈ $C_{22}H_{46}$	50,7	18,8
HANSTEIN [40*b*]	$C_{16}H_{34}$*	49,0	18,2
BRICKWEDDE *et al.* [38]	H_2O	42,2 ± 1,8	—
LIBBY and LONG [39]	CH_4	36 ± 4	—
	C_2H_4	46 ± 3	17,1 ± 1,1
	C_2H_6	46 ± 3	17,1 ± 1,1
GIBERT and ROSSEL [46]	H_2	31,6 ± 0,6	—
ROSSEL [47]	H_2O	40,7 ± 1,5	—
CARROLL [41]	H_2	31,8 ± 0,5	—
	H_2O	44,6 ± 0,5	—
	CH_4	45,4 ± 0,3	—
	C_2H_6	46,4 ± 0,5	17,4 ± 0,2
	C_3H_8	46,9 ± 0,6	17,6 ± 0,2
	C_4H_{10}	48,7 ± 0,6	18,3 ± 0,2
	$n\,C_{16}H_{34}$ solid	44,9 ± 0,4	16,8 ± 0,15
	$n\,C_{16}H_{34}$ gas	50,1 ± 0,4	18,8 ± 0,15
	≈ $C_{22}H_{46}$	49,8 ± 0,2	18,7 ± 0,1
	$n\,C_{32}H_{66}$	50,2 ± 0,2	18,8 ± 0,1
MARSHALL [46] **	H_2O	44,4	—

* Thick In detectors covered with In foils to minimize resonance activation were used.

** Neutrons from uranium-graphite pile.

cross-section for H is calculated from the molecular cross-section by assuming additivity of the atomic cross-sections (incoherent scattering) *.

* This additivity may break down for thermal neutrons, owing to interference effects, especially if the scatterer is in a crystalline or polycrystalline state. See on this point H. BEYER and M. WHITAKER, *P.R.* **57**, 976. 1940; F. RASETTI, *P.R.* **58**, 321. 1940; O. HALPERN, M. HAMERMESH and H. JOHNSON, *P.R.* **59**, 981. 1941; R. SEEGER and E. TELLER, *P.R.* **62**, 37. 1942; R. WEINSTOCK, *P.R.* **65**, 1. 1944; E. FERMI, W. STURM and R. SACHS, *P.R.* **71**, 589. 1947.

It is therefore advisable to measure also the cross-section of C or O, as the case may be, under the same conditions. (In the experiments reported below, this has always been done, unless otherwise indicated.) The experiments cover an extended range of energies of the impinging neutrons; the energy regions explored are actually determined by the available neutron sources. At the moment, we shall be concerned with the region of *slow neutrons*, up to about 50 eV, within which the scattering cross-section will not be expected to vary appreciably. The value obtained can therefore be regarded as the limiting cross-section for zero energy.

Numerous measurements have been carried out with *"thermal" neutrons*, i.e. neutrons in thermal equilibrium with the material from which they emerge; if T denotes the absolute temperature of this material, their mean kinetic energy is of the order of kT, i.e. $\approx$ 0,026 eV at room temperature. Until recently, the usual source of thermal neutrons, or neutron "howitzer", has been a Rn–Be tube surrounded with paraffin. But some experiments (MARSHALL [46]) have already been performed with the thermal neutrons issuing from a uranium-graphite pile. The thermal neutrons are selected by comparing the measurements with and without interposition of a Cd filter. The neutron beam is collimated by suitable walls and diaphragms lined with B and Cd, and detected by means of a BF_3 ionization chamber connected with a linear amplifier. The cross-sections obtained must be corrected for the effect of proton binding (*6.212, 6.213*). We shall first give results of measurements in which the paraffin of the source as well as the scatterer were at room temperature. In the table *6.22*–1, we have collected — together with some cross-section values obtained with lighter molecules — chiefly those pertaining to hydrocarbons of high molecular weight; only in the latter case has the reduction factor from the cross-section S' per bound proton to the free proton cross-section S been worked out with any accuracy (ARLEY [38], p. 31, 32).

If the source is cooled (by liquid air, say) a marked increase of the cross-section S' is observed, in agreement with the theoretical expectation. For the ratio of the new cross-section to that at room temperature, it is found:

6.22–2. *Scattering of cooled neutrons by paraffin*

Reference	Neutron temperature	Ratio observed	Ratio theoretical
FRISCH *et al.* [37]	$\approx$ 88 °K	1,09 $\pm$ 0,05	1,34
POWERS *et al.* [38]	$\approx$ 120 °K	1,3 $\pm$ 0,06	1,25

No better agreement can be expected in view of the far-reaching schematization of Arley's theory.

More satisfactory results, however, have been obtained by investigating the scattering of neutrons of ordinary thermal velocity by hydrogen gas cooled to very low temperatures; for the reduction formula of SACHS and

TELLER [41] may then be applied to derive from the observed cross-section $\mathcal{S}'$ the free proton value $\mathcal{S}$. The following table summarizes the relevant data:

6.22–2a. *Scattering of thermal neutrons by cooled hydrogen gas*

Reference	Gas temperature	$\mathcal{S}'$ 10^{-24} cm²	$\mathcal{S}$ 10^{-24} cm²
LIBBY and LONG [39]	90 °K	22 ± 4	14,7 ± 2,7
GIBERT and ROSSEL [46]	77 °K	27,2 ± 1,6	19,2 ± 1,1
	20 °K	26,2 ± 2,9	20,3 ± 2,3

The large and rather uncertain correction necessitated by the binding effect makes it desirable to carry out measurements with neutrons of energy $> \approx 1$ eV, when this effect becomes negligible. As already stated, the cross-section for neutrons of a few volts may, of course, equally well as that for thermal neutrons, be identified with its limiting value for zero energy: for the other energy quantities which enter into the description of the scattering process are of "nuclear" order of magnitude (0,1 ... 1 MeV). Now, the scattering of *resonance neutrons*, the energies of which are just in the region from 1 to about perhaps 50 eV, may be

6.22–3. *Scattering of resonance neutrons by protons*

Reference	Source	Filter	$\mathcal{S}$ 10^{-24} cm²
SIMONS [40]	Be (γ_{Ra}, n) in paraffin	Ag, I	14,4 ± 0,8 *
COHEN *et al.* [39]	id.	Rh	20 ± 2 **
COHEN *et al.* [40]	id.	Ag, In	
HANSTEIN [40*b*]	Be (p, n), beam collimated	In	21 ± 1
HANSTEIN [41]	id.	In, I	

* In deriving this value, large corrections had to be applied owing to the unfavourable "geometry" of the arrangement. Moreover, the value here given differs from that published by Simons, because a better value of the oxygen cross-section has been used in reducing the experimental results. According to CARROLL [41], the slow neutron cross-section of oxygen is

$$\mathcal{S}_O = (4{,}12 \pm 0{,}10) \cdot 10^{-24} \text{ cm}^2;$$

BRICKWEDDE *et al.* [38] give essentially the same value.

** In the reduction of the data, the slow neutron scattering cross-section of carbon

$$\mathcal{S}_C = 4{,}83 \cdot 10^{-24} \text{ cm}^2.$$

as given by GOLDHABER and BRIGGS [37], has been adopted; it agrees with the more recent measurements of HANSTEIN and DUNNING [40*a*] and MARSHALL [46].

studied thanks to a "resonance filter" method, devised independently by COHEN, GOLDSMITH and SCHWINGER [39] and by L. SIMONS [40]. A filter of a material possessing a resonance level for slow neutrons is placed before the scatterer, and the detector is made of the same material (shielded by Cd to eliminate thermal neutrons); measurements are carried out with and without the filter. The same source as for thermal neutrons can be used; but larger intensities have been obtained by means of the reaction Be (p, n), with a cyclotron as the source of the necessary protons of high energy (threshold of the reaction at 2,01 MeV) (HANSTEIN and DUNNING [40*a*]).

From table *6.22–3* it may safely be concluded that SIMONS' [40] value is definitely too low; in particular, it is not (as was for some time believed owing to an overestimation of the proton binding effect) in agreement with the thermal neutron measurements. It must also be observed, in this connexion, that systematic errors in such experiments will tend to lower the observed value of the scattering cross-section. There is no doubt that the most reliable figure is that given by HANSTEIN [40*b*, 41]; we shall therefore adopt, for the scattering cross-section for neutrons of zero energy, the value

$$\mathcal{S} = (21 \pm 1) \cdot 10^{-24}\ \mathrm{cm}^2. \tag{32}$$

6.23. *The ^1S level of small energy of the deuteron.* Let us now compare the value (32) of the scattering cross-section for zero energy neutrons with the theoretical result. If there were no spin-dependent term of nuclear interaction, the singlet and triplet S levels would practically coincide and the quantity in question would be just four times the limiting value $\mathcal{S}^{(0)}$ given by (13); i.e., disregarding the finite range correction (which does not alter the order of magnitude) and taking account of (14), $4\pi\,(^3a_0)^2 = 2{,}39 \cdot 10^{-24}\ \mathrm{cm}^2$. This figure is in complete disagreement with the experimental result (32), which is about 9 times larger. For larger neutron energies, on the other hand, the discrepancy would soon become very much smaller: according to (12) and (20), one would have (again neglecting the finite range correction)

$$\mathcal{S}_{\text{no spin force}} = \frac{4\pi\hbar^2}{M} \cdot \frac{1}{\frac{1}{2}E + |\varepsilon_0|};$$

for $E = 4{,}1$ MeV, say, this gives $1{,}24 \cdot 10^{-24}\ \mathrm{cm}^2$, indeed of the same order of magnitude as the measured value $1{,}73 \cdot 10^{-24}\ \mathrm{cm}^2$ (table *6.413*).

The way out of this difficulty, as first suggested by WIGNER (quoted by B & B, p. 117), is just to assume that owing to a *strong spin dependence of the nuclear potential,* there is some (actual or virtual) ^1S level of much smaller energy (in absolute value) than the ground state. In fact, the cross-section (for zero range) then becomes (writing, for symmetry, $|\,^3\varepsilon^{(0)}\,|$

instead of $|\varepsilon_0|$ for the energy of the ground state)

$$\mathcal{S}(\mathsf{E}) = \frac{\pi\hbar^2}{M}\left[\frac{3}{\frac{1}{2}\mathsf{E} + |{}^3\varepsilon^{(0)}|} + \frac{1}{\frac{1}{2}\mathsf{E} + |{}^1\varepsilon^{(0)}|}\right] \qquad \text{(zero range)}, \tag{33}$$

and for $\mathsf{E} = 0$,

$$\mathcal{S}(0) = 3\pi\,({}^3a_0)^2 + \pi\,({}^1a_0)^2 \qquad \text{(zero range)}$$

$${}^3a_0 = \frac{\hbar}{\sqrt{M\,|\,{}^3\varepsilon^{(0)}|}} = 0{,}44\cdot 10^{-12}\,\text{cm}, \qquad {}^1a_0 = \pm\frac{\hbar}{\sqrt{M\,|\,{}^1\varepsilon^{(0)}|}}. \tag{34}$$

While this modification has but small effect for relatively large values of E, it gives much larger results for very small energies if $|\,{}^1\varepsilon^{(0)}| \ll |\,{}^3\varepsilon^{(0)}|$, the contribution to the scattering from the singlet term then becoming preponderant. Adjusting the parameter $|\,{}^1\varepsilon^{(0)}|$ so as to get for $\mathcal{S}(0)$ Hanstein's value (32), we get from (34)

$${}^1a_0| = (2{,}47 \pm 0{,}07)\cdot 10^{-12}\,\text{cm}, \quad |\,{}^1\varepsilon_0| = 0{,}067 \pm 0{,}004\,\text{MeV}\,\text{(zero range)}; \tag{35}$$

we shall see presently that the correction for finite range does not appreciably modify these values. The analysis of the scattering data yields only the absolute value of the quantities ${}^1a_0$ and ${}^1\varepsilon_0$, so that the question whether the ${}^1S$ level is actual or virtual cannot be decided on this evidence.

6.231. *Radiative capture of slow neutrons.* The large spin dependence of the proton-neutron force revealed by the strong scattering of slow neutrons by protons is further confirmed by the discussion of the radiative capture of such neutrons by protons, leading to the formation of deuterons with emission of γ-rays. This point will be treated together with the other radiative processes in a later section (*6.52*). It must be pointed out that in contrast to the elastic scattering, the radiative capture effect is sensitive to the actual or virtual character of the ${}^1S$ level; however, as we shall see, the evidence in this respect is not entirely conclusive. A definitive settlement of the question is provided by experiments on neutron scattering by para- and ortho-hydrogen, to the study of which we shall now proceed.

6.3. Slow neutron scattering by hydrogen molecules

6.30. The scattering of very slow neutrons by molecular hydrogen requires special consideration when the energy of the neutrons becomes of the order of magnitude of, or lower than, the excitation energy of the rotation levels of the molecule; such a kinetic energy ($\approx$ 0,01 eV) corresponds, of course, to a de Broglie wave-length of the order of magnitude of the mean distance between the two H atoms. We must then take account of two peculiarities, which, as emphasized by TELLER [36], have a considerable bearing on the problem of the spin dependence of the proton-neutron interaction and in particular on the question whether the ${}^1S$ level of low energy of the deuteron is an actual or virtual one. In the first place, hydrogen molecules provide us with two different arrangements

of the proton spins, viz. the para-configuration (antiparallel spins) and the ortho-configuration (parallel spins). If the nuclear forces did not depend on spin, this would of course make no difference to the scattering of neutrons *; but if they are spin-dependent, the two configurations will react differently. Assume, in fact, (to consider a simple case) that the temperature of the hydrogen is so low that practically all molecules are in the lowest para- or ortho-configuration (in some given proportion). If the neutron energy is too small to excite a para-molecule into the ortho-state, the scattering by para-hydrogen will only involve para-para transitions in the molecule, while the scattering by ortho-hydrogen will comprise both ortho-ortho and ortho-para transitions, the latter being entirely brought about by the spin dependent interaction between neutron and proton. Further, since the wave-length of the neutrons is of the same order of magnitude as the distance between the protons, we shall expect strong interference of the neutron waves scattered from the two protons. The effect of such an interference will depend very sensitively on the relative sign of the triplet and singlet phase constants, i.e. of the constants ^3a and ^1a; we thus see that, while the scattering by protons only yields information on the absolute values of these constants, the phenomenon under discussion may reveal whether the ^1S-level connected with ^1a is an actual or a virtual one.

6.31. *Experiments.* The effects just described have actually been found when looked for. The chief difficulty in these experiments is to get rid of any admixture of neutrons of higher velocity which tend to blur the characteristic difference sought. If one uses as a neutron source the usual "howitzer" (*6.22*), the paraffin being cooled, say, to liquid air temperature, the Cd filter procedure becomes insufficient, since it only suppresses neutrons of energy above 0,4 eV. Still, the first experiments, carried out with this source (HALPERN *et al.* [37], BRICKWEDDE *et al.* [38], LIBBY and LONG [39]), were already able to establish the essential qualitative conclusions that

(1) $o\,H_2$ scatters neutrons of $\approx$ 0,01 eV mean energy ($\approx$ 120 °K temperature) much more than $p\,H_2$;

(2) the cross-section of $p\,H_2$ is markedly smaller for neutrons of $\approx$ 0,01 eV mean energy than for ordinary ($\approx$ 300 °K thermal neutrons.

The quantitative results for liquid hydrogen (BRICKWEDDE *et al.* [38]) and for hydrogen gas cooled down to $\approx$ 90 °K (LIBBY and LONG [39]) were as follows:

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* Except in so far as the spacing of the rotational levels is different in ortho- and para-hydrogen. This effect would only be noticeable for a critical energy of the impinging neutrons. Cf. SCHWINGER and TELLER [*37a*].

6.31. *Scattering of neutrons by molecular hydrogen*

Neutron temperature		≈ 300 °K	≈ 120 °K	"*p*-filtered" 120 °K
Molecular cross-section * (10^{-24} cm²)	o H_2	55,8 (44 ± 8)	78,7 (78 ± 10)	100,1
	p H_2	29,0 (44 ± 8)	17,6 (38 ± 8)	12,6

* The figures between brackets refer to hydrogen gas at ≈ 90 °K.

The figures given in this table for liquid hydrogen correspond to a molecular cross-section of $48{,}7 \cdot 10^{-24}$ cm² for scattering of ordinary thermal neutrons by liquid hydrogen of normal composition (75 % o H_2); the comparison of this value with that for gaseous hydrogen at the same temperature of 20 °K, viz. $(52{,}4 \pm 5{,}8) \cdot 10^{-24}$ cm² (table *6.22–2a*), shows that there is no appreciable effect of intermolecular forces. Libby and Long's values for hydrogen gas at 90 °K, quoted in the above table, point, though less convincingly, to the same conclusion, when allowance is made for the thermal motion of the gas molecules. The last column in table *6.31* concerns 120 °K neutrons which have been filtered by a layer of liquid para-hydrogen; owing to the relative transparency of pH_2 to "cold" neutrons, the effect of this filtering is to attenuate the high energy "tail" of the neutron distribution. The large differences caused by such filtering strikingly illustrate the disturbing influence of the inhomogeneity of the neutrons.

A considerable improvement in this respect could be obtained with the help of the neutron monochromator devised by ALVAREZ [38]. The principle of this apparatus is very simple: it consists in modulating the ion beam which produces neutrons by some suitable nuclear reaction, and arranging the linear amplifier of the detecting BF_3 ionization chamber to be sensitive only when the ion source is off. This arrangement will only record neutrons of velocity d/t, d being the distance of the chamber from the source and t the phase difference in time between the modulations of the beam and the amplifier. In the actual apparatus (ALVAREZ and PITZER [40*b*]), the deuteron beam from a cyclotron activated a Be target and the Be(d, n) neutrons were slowed down by a paraffin block placed near the target *. The distance d was about 7 m; the modulation frequency being $60 \cdot \text{sec}^{-1}$, the time t could be chosen so as to select neutrons of ≈ 20 °K (the corresponding effective velocity of these neutrons ** being $\sqrt{\pi kT/2M} \approx 509$ m/sec, this gives $t \approx 1/73$ sec). The scatterer was hydrogen gas cooled by boiling liquid hydrogen at 20,4 °K. After correction for capture of the neutrons by the protons, the scattering cross-sections obtained were

$$\text{for} \begin{cases} o\,H_2 : 100 \pm 3 \\ p\,H_2 : 5{,}2 \pm 0{,}6 \end{cases} \cdot 10^{-24}\ \text{cm}^2, \qquad (1)$$

* The time t must then be sufficiently long compared with the life-time of the neutrons in paraffin, to avoid any disturbance from the interposition of the paraffin block.

** This is the effective velocity for B absorption, cf. BETHE [37], § 60A.

thus qualitatively agreeing with the former measurements, but yielding a much larger value of the ratio ortho/para.

6.32. *Theory.* The quantitative treatment of the effect under discussion (SCHWINGER and TELLER [37a]) requires only a slight generalization of the procedure sketched in *6.213*. We have just to replace in (*6.213*–29) the functions φ_0, φ_1 by the required eigenfunctions of the hydrogen molecule (including the motion of its centre of gravity) and the operator $\mathcal{U}$ by

$$\overline{\mathcal{U}} = \mathcal{U}(|\vec{x}^{(n)} - \vec{x}^{(p_1)}|) + \mathcal{U}(|\vec{x}^{(n)} - \vec{x}^{(p_2)}|). \tag{2}$$

Separating $\overline{\mathcal{U}}$ into two parts, symmetrical and antisymmetrical, respectively, in the proton spins, we get, by (*6.213*–28) [with $\delta_{r_0}(r) \to \delta(\vec{x})$] and (*6.213*–31),

$$\begin{aligned}\overline{\mathcal{U}} = \frac{\pi\hbar^2}{M}\Big\{(3\cdot{}^3a + {}^1a) + ({}^3a - {}^1a)\,\vec{\sigma}^{(n)}\cdot\frac{\vec{\sigma}^{(p_1)} + \vec{\sigma}^{(p_2)}}{2}\Big\}\cdot[\delta(\vec{x}^{(np_1)}) + \delta(\vec{x}^{(np_2)})] \\ + \frac{\pi\hbar^2}{M}({}^3a - {}^1a)\,\vec{\sigma}^{(n)}\cdot\frac{\vec{\sigma}^{(p_1)} - \vec{\sigma}^{(p_2)}}{2}[\delta(\vec{x}^{(np_1)}) - \delta(\vec{x}^{(np_2)})],\end{aligned} \tag{3}$$

with $\vec{x}^{(np_l)} \equiv \vec{x}^{(n)} - \vec{x}^{(p_l)}$. From this expression, it is easily deduced that the scattering cross-section for transitions involving no change of the total spin $\frac{1}{2}[\vec{\sigma}^{(p_1)} + \vec{\sigma}^{(p_2)}]$ of the molecule (i.e. para-para and ortho-ortho transitions) contains the factor

$$(3\cdot{}^3a + {}^1a)^2 + S(S+1)({}^3a - {}^1a)^2, \tag{4}$$

S being the spin quantum-number of the molecular state ($S = 0$ for para-states, $S = 1$ for ortho-states). For transitions involving a change of spin (ortho-para and para-ortho transitions), the cross-section is proportional to

$$({}^3a - {}^1a)^2\,\frac{3}{2S+1}, \tag{5}$$

S denoting the spin quantum-number of the initial state (that of the final state being 1—S). These expressions exhibit very clearly the interference features described above (*6.30*).

For the cross-sections of para- and ortho-hydrogen in their lowest states, we can write accordingly

$$\begin{aligned}\mathcal{S}_{\text{para}} &= \mathcal{S}_{p\to p}(3\cdot{}^3a + {}^1a)^2 + \mathcal{S}_{p\to o}({}^3a - {}^1a)^2 \\ \mathcal{S}_{\text{ortho}} &= \mathcal{S}_{o\to o}(3\cdot{}^3a + {}^1a)^2 + (2\mathcal{S}_{o\to o} + \mathcal{S}_{o\to p})({}^3a - {}^1a)^2,\end{aligned} \tag{6}$$

the coefficients $\mathcal{S}$ being certain functions of the neutron energy, represented by * fig. *6.32*; the indices correspond to the transition involved: $\mathcal{S}_{p\to p}$ and

* Analytical expressions for the $\mathcal{S}$'s, which, besides the neutron energy, involve only parameters of the hydrogen molecule, are given by SCHWINGER and TELLER [37a]. A

$\mathcal{S}_{o\to o}$ pertain to elastic scattering from the lowest para- or ortho-state, respectively, while $\mathcal{S}_{o\to p}$ and $\mathcal{S}_{p\to o}$ refer to inelastic scattering processes. The latter ($\mathcal{S}_{p\to o}$) is, of course, only different from zero for neutron energies larger than the excitation energy of the lowest ortho-level, viz. 0,023 eV. As to $\mathcal{S}_{o\to p}$, it varies as $1/v$ at low velocities and remains practically constant for energies higher than about 0,01 eV.

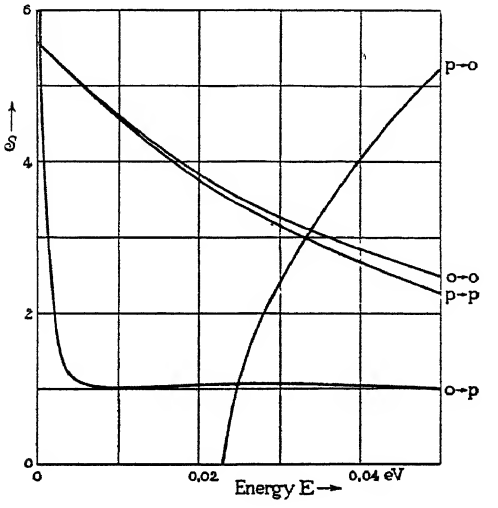

Fig. 6.32. Coefficients of the neutron scattering cross-sections of para- and ortho-hydrogen, in terms of the neutron energy E.

The states of the deuterium molecule D_2 can also be divided into para- and ortho-configurations according to their parity; in this case, the para-states correspond to a molecular spin $S = 1$ and odd values of the total angular momentum J, the ortho-states to $S = 0$ or 2 and even values of J. In view of possible experiments with deuterium, the theory of slow neutron scattering has been worked out for this case by HAMERMESH and SCHWINGER [46]. It proceeds on exactly the same lines as above, starting from the expression (*6.213–31c, 31d*) for the equivalent scattering potential of the deuteron.

6.33. *Analysis of experiments.* Let us now, by way of illustration, take ${}^1a/{}^3a = \pm 4$, — an order of magnitude suggested by (*6.23–34,35*). From formula (6), together with figure *6.32*, we see that, according as the ${}^1S$ level of low energy of the deuteron is actual or virtual, the ratio

$$(3\cdot{}^3a + {}^1a)^2/({}^3a - {}^1a)^2$$

will change from 5,44 to 0,11. In the first case, the elastic scattering would predominate, so that (considering the properties of the coefficients $\mathcal{S}$) the ortho and para cross-sections would always be of the same order of

more precise calculation of effective cross-sections, taking account of the Maxwellian contribution of the velocities of the molecules, is carried out by SCHWINGER [40]; in this last paper, the expression for $\mathcal{S}_{o\to p}$ actually utilized differs from that given by SCHWINGER and TELLER by a factor $\frac{1}{3}$, which had been omitted by these authors.

magnitude, and the values for cold neutrons would be only moderately larger than those for ordinary thermal neutrons. In the case of a virtual ^1S level, however, in which the inelastic processes are enhanced, the para cross-section will be very much smaller for cold than for thermal neutrons, owing to the vanishing in the former energy range of the coefficient $\mathcal{S}_{p\to o}$; again, for cold neutrons, the para cross-section will for the same reason be very much smaller than the ortho cross-section. It has been stressed above (*6.31*) that both these predictions are fully confirmed by all experiments, which conclusively establishes that the ^1S *level is a virtual one.*

The measurements by Alvarez, given by (1), allow of a more quantitative analysis (SCHWINGER [40]). For the neutron energy $E \approx kT \approx 0{,}00173$ eV, one gets after correcting for the thermal motion of the molecules

$$\mathcal{S}_{p\to p} = 6{,}473, \quad \mathcal{S}_{o\to o} = 6{,}291, \quad \mathcal{S}_{o\to p} = 1{,}447,$$

while $\mathcal{S}_{p\to o} = 0$. According to (6), the para cross-section gives directly

$$(3\cdot{}^3a + {}^1a)^2 = (0{,}803 \pm 0{,}092)\cdot 10^{-24}\,\mathrm{cm}^2; \tag{7}$$

the ortho cross-section then yields

$$({}^3a - {}^1a)^2 = (6{,}77 \pm 0{,}26)\cdot 10^{-24}\,\mathrm{cm}^2. \tag{8}$$

These numbers are not in very good agreement with the proton scattering cross-section for neutrons of zero energy, given by (*6.22–32*); in fact, the latter quantity may be written

$$\mathcal{S}(0) = 3\pi\,({}^3a)^2 + \pi\,({}^1a)^2 = \frac{\pi}{4}\,[(3\cdot{}^3a + {}^1a)^2 + 3\,({}^3a - {}^1a)^2], \tag{9}$$

which would give only $\mathcal{S}(0) = (16{,}57 \pm 0{,}69)\cdot 10^{-24}$ cm^2. It is therefore desirable to repeat the measurements: especially the quantity $(3\cdot{}^3a + {}^1a)^2$, immediately furnished by the para cross-section, is very sensitive to the value of ^3a, i.e. — as appears from (*6.211–17*) — to the range of the nuclear force *.

6.34. *The neutron spin.* - The general evidence of scattering of slow neutrons incidentally provides us, as pointed out by SCHWINGER [*37b*], with an indirect proof that the neutron spin is $\frac{1}{2}$ rather than, say, $\frac{3}{2}$. If the neutron has an arbitrary intrinsic momentum $\vec{S}^{(n)}$, and if S_n denotes the

* If we try to calculate the a's from (7) and (8), we get — taking account of the fact that ^3a and ^1a must be of opposite signs — two systems of solutions:

	^3a (10^{-12} cm)	^1a (10^{-12} cm)
I	$0{,}874 \pm 0{,}025$	$1{,}728 \pm 0{,}025$
II	$0{,}426 \pm 0{,}025$	$2{,}176 \pm 0{,}025$

According to (*6.211–16, 17, 18*) and (*6.21–14*), the corresponding widths of a potential well yielding such values of ^3a are $\approx 0{,}8\cdot 10^{-12}$ cm and ≈ 0, respectively. Neither of these range values is of a reasonable order of magnitude. Cf. also HAMERMESH and SCHWINGER [47].

corresponding quantum number, we must (*6.213–31a, b*) introduce two amplitudes ${}^{(2S_n)}a$, ${}^{(2S_n+2)}a$, in terms of which the scattering cross-section of protons for very slow neutrons may be written

$$\mathcal{S} = 4\pi \left[\frac{S_n+1}{2S_n+1} \cdot {}^{(2S_n+2)}a^2 + \frac{S_n}{2S_n+1} \cdot {}^{(2S_n)}a^2 \right], \tag{10}$$

while the scattering radius takes the form

$$a = \left[\frac{S_n+1}{2S_n+1} \cdot {}^{(2S_n+2)}a + \frac{S_n}{2S_n+1} \cdot {}^{(2S_n)}a \right] + \frac{1}{2S_n+1} \left[{}^{(2S_n+2)}a - {}^{(2S_n)}a \right] \vec{S}^{(n)} \vec{\sigma}^{(p)}. \tag{11}$$

The factors (4), (5) then become

$$(S \to S') \qquad \left[\frac{S_n+1}{2S_n+1} \cdot {}^{(2S_n+2)}a + \frac{S_n}{2S_n+1} \cdot {}^{(2S_n)}a \right]^2 + \frac{S(S+1)\, S_n(S_n+1)}{3(2S_n+1)^2} \left[{}^{(2S_n+2)}a - {}^{(2S_n)}a \right]^2 \tag{12}$$

$$(S \to 1-S) \qquad \left[{}^{(2S_n+2)}a - {}^{(2S_n)}a \right]^2 \cdot \frac{S_n(S_n+1)}{3(2S_n+1)^2} \cdot \frac{3}{2S+1}.$$

If, e.g., the neutron spin were $\frac{3}{2}$, the excited state of the deuteron would be a quintet state, and (10) would give

$$\mathcal{S} = 4\pi \left[\tfrac{3}{8} ({}^3a)^2 + \tfrac{5}{8} ({}^5a)^2 \right]; \tag{13}$$

from this formula, we can derive $|{}^5a|$ and the energy $|{}^5\varepsilon^{(0)}|$ of the corresponding actual or virtual quintet level. If we apply the finite range correction (*6.211*) on the assumption of a potential well of given width, we find, by a procedure explained in detail in the next section (*6.431*), the following results:

6.34. *Scattering radii and excited deuteron level for neutron spin* $\frac{3}{2}$

D	${}^3a$	$\lvert {}^5a \rvert$	$\lvert {}^5a/{}^3a \rvert$	$\lvert {}^5\varepsilon^{(0)} \rvert$ (actual level)	${}^5\varepsilon^{(0)}$ (virtual level)
10^{-13} cm	10^{-12} cm	10^{-12} cm		MeV	MeV
1,94	0,538	1,58	2,94	0,19	0,14
2,8	0,585	1,57	2,68	0,20	0,14

On the other hand, according to (12) and (6), the scattering cross-sections of para- and ortho-hydrogen for cold neutrons may be written

$$\begin{aligned} \mathcal{S}_{\text{para}} &= \tfrac{1}{4} \mathcal{S}_{p\to p} (3 \cdot {}^3a + 5 \cdot {}^5a)^2 \\ \mathcal{S}_{\text{ortho}} &= \tfrac{1}{4} \mathcal{S}_{o\to o} (3 \cdot {}^3a + 5 \cdot {}^5a)^2 + \tfrac{5}{4} (2\mathcal{S}_{o\to o} + \mathcal{S}_{o\to p}) ({}^3a - {}^5a)^2, \end{aligned} \tag{14}$$

the coefficients $\mathcal{S}$ being the same as in the case of spin $\frac{1}{2}$. With the above values of the ratio $|{}^5a/{}^3a|$, it will be seen that the ratio $\mathcal{S}_{\text{ortho}}/\mathcal{S}_{\text{para}}$ is of the

order of magnitude 1 if the ^5S level is actual, or 2 ... 3 if it is virtual; in either case in sharp contradiction with experiment *.

6.35. *Scattering of polarized neutrons by protons.* In view of the possibility of producing partially polarized neutron beams **, it is interesting to note that, as pointed out by SCHWINGER and RABI [37c], the effect of the scattering of such a beam by (unpolarized) protons on its polarization affords a quite sensitive means of testing the spin dependence of the nuclear force and the nature of the excited ^1S level of the deuteron. No experiments on these lines, however, have as yet been attempted.

In a general way, the spin dependence of the wave-function representing the incident neutron (1) and proton (2) may be expressed as follows:

$$(\alpha^{(1)} v_+(1) + \beta^{(1)} v_-(1))\,(\alpha^{(2)} v_+(2) + \beta^{(2)} v_-(2))\,,$$

$v_\pm$ being the eigenfunctions of σ_z. The polarization p of the incident neutron is defined by

$$p = \frac{|\alpha^{(1)}|^2 - |\beta^{(1)}|^2}{|\alpha^{(1)}|^2 + |\beta^{(1)}|^2}, \tag{15}$$

while for the amplitudes of the unpolarized protons $|\alpha^{(2)}|^2 = |\beta^{(2)}|^2$. It is then easily seen from (*6.21–9*) that the absolute squares of the amplitudes corresponding to the two spin orientations of the scattered neutron wave (restricted to its S-component) are proportional to

$$|\alpha^{(1)}|^2 \cdot [|^3\Delta|^2 + \tfrac{1}{4}|^3\Delta + {}^1\Delta|^2] + |\beta^{(1)}|^2 \cdot \tfrac{1}{4}|^3\Delta - {}^1\Delta|^2$$

and

$$|\beta^{(1)}|^2 \cdot [|^3\Delta|^2 + \tfrac{1}{4}|^3\Delta + {}^1\Delta|^2] + |\alpha^{(1)}|^2 \cdot \tfrac{1}{4}|^3\Delta - {}^1\Delta|^2$$

respectively; we have put $^\sigma\Delta = e^{2i\,^\sigma\delta^{(0)}} - 1$. Hence the polarization of the scattered neutron wave is found to be

$$p_{\text{scatt}} = p\left[1 - \frac{\sin^2({}^3\delta^{(0)} - {}^1\delta^{(0)})}{\sin^2{}^1\delta^{(0)} + 3\sin^2{}^3\delta^{(0)}}\right]. \tag{16}$$

The scattering thus causes a change of polarization of the incident neutron beam whenever the potential is spin dependent. The sensitiveness of the effect can be judged from the case of very small neutron energies. According to (*5.32–21*), formula (16) then becomes

$$p_{\text{scatt}} = p\left[1 - \frac{({}^3a - {}^1a)^2}{({}^1a)^2 + 3\,({}^3a)^2}\right]; \tag{17}$$

this result follows also directly from (*6.213–29, 30, 31*) and is therefore

* On the contrary, no clear-cut evidence against the spin $\frac{3}{2}$ can be drawn from the proton scattering cross-section for neutrons of 2,5 MeV, as was wrongly asserted by H. GOLDSMITH and L. MOTZ, *P.R.* 53, 947. 1938. For this energy, assuming the ^5S-level to be virtual, we get in fact a cross-section

$$2{,}68 \cdot 10^{-24}\,\text{cm}^2 \text{ for } D = 1{,}94 \cdot 10^{-13}\,\text{cm}$$
$$\text{or } 2{,}80 \cdot 10^{-24}\,\text{cm}^2 \text{ for } D = 2{,}8 \cdot 10^{-13}\,\text{cm}$$

of the same order of magnitude as the observed one (table *6.412*).

** According to a recent remark by SCHWINGER [46], the resonance scattering of neutrons by Helium (*17.42*) should have a strong polarizing effect.

valid also for bound protons. For $|\,^1a/^3a\,| = 4$ one gets

$$\begin{aligned} &^1S \text{ level actual: } \; p_{\text{scatt}}/p \approx \;\; 0{,}53 \\ &^1S \text{ level virtual: } \; p_{\text{scatt}}/p \approx -0{,}32. \end{aligned} \tag{18}$$

6.4. Scattering of fast neutrons by protons

6.40. We may now proceed to the study of neutrons of higher energy (from, say, 0,1 MeV), which, for brevity, may be termed "fast". In this section, we shall first collect all the relevant experimental evidence. This comprises, on the one hand, numerous measurements of the total scattering cross-section of protons for fast neutrons of various energies, and, on the other hand, some results concerning the angular distribution of the scattered neutrons (or of the recoil protons), likewise for different energies of the incident particles. The theoretical discussion will be limited, however, to an analysis of the dependence of the total cross-section on the energy of the incident neutrons, in so far as it can be accounted for by the S-phase only, i.e. in so far as it depends on the effective potentials for even states. The discussion of the contributions from higher phases both to the total cross-section and to the departure of the angular distribution of the scattered particles from isotropy (in the barycentric system) necessitates some knowledge of the effective potentials for odd states as well, and will conveniently be postponed to a later section (*8.33*).

6.41. *Total scattering cross-section of protons for fast neutrons.* When experimenting with "fast" neutrons, the chief difficulty is to get sufficiently homogeneous neutron beams of different energies *. In successive intervals of increasing energy, various sources and methods of detection have to be used for that purpose. We shall begin with a survey of the determinations of the total cross-section, treating these different energy regions separately.

6.411. *Scattering of neutrons of energies 0,1 ... 1 MeV.* In the interval 0.1 ... 1 MeV, one may use photoneutrons from Be or D activated by the γ-rays of ThC″ (2,62 MeV) or RaC (mainly 1,76 MeV); the thresholds of the reactions $D(\gamma, n)$, $Be(\gamma, n)$ are in fact 2,18 and 1,63 MeV, respectively. The reaction $^{12}C(d, n)$, with deuterons from a high voltage tube, also yields low energy neutrons, the energy balance of the (endothermic) reaction being —0,27 MeV. Another convenient neutron source in the energy interval considered is provided by the $^7Li(p, n)$ reaction. The neutrons can be detected either in an ionization chamber (I.C.) filled with hydrogen under pressure, or by the activation of a suitable element surrounded with paraffin or water. In this way, the values of the cross-section listed in table *6.411* have been obtained; the older ones are not very accurate **.

* In the following survey, we shall omit a few measurements in which the neutron energy was not sufficiently well-defined, or which for some reason do not seem quite reliable. They are listed in the very complete tables of DIEBNER, HERRMANN and GRASSMANN [42].

** In tables *6.411* and *6.412*, the values of the neutron energies have been recalculated from the data about the nuclear reactions mentioned in the text.

6.411. *Scattering of neutrons of 0,1 ... 1 MeV energy by protons*

Reference	Source	Detector	Energy (approxim.) MeV	S 10^{-24} cm²
LEIPUNSKI *et al.* [36 *b*]	Be(γ_{Ra},*n*)	Ag in water	0,12	8 ±2 *
GOLOBORODKO and LEIPUNSKI [39]	D(γ_{RdTh},*n*)	Dy in paraffin	0,22	5,0 ±1,0
AMALDI *et al.* [40]	^{12}C(*d*,*n*)	I.C. with 20 atm. H_2; electrometer	0,4	8,7
GOOD and SCHARFF-GOLDHABER [40]	Be(γ_{RdTh},*n*)	I.C. with 5 atm. H_2; linear amplifier	0,9	3,70+0,35
FRISCH [46]	^{7}Li(*p*,*n*)	I.C. with 0,55 to 2,65 atm. H_2; linear amplifier	0,035	16,74±0,41
			0,095	13,46±0,39
			0,265	9,12±0,24
			0,490	6,33±0,21
BAILEY *et al.* [46]	^{7}Li(*p*,*n*)	I.C. with 4 atm. methane; linear amplifier	0,35	7,15±0,24
			0,46	6,52±0,15
			0,72	5,22±0,12
			0,97	4,45±0,08
			1,0 ±0,1	4,16±0,15
	^{12}C(*d*,*n*)	argon filled I.C. with paraffin radiator	1,6	3,36±0,08
			2,0	2,96±0,07

* Value calculated with the help of the C cross-section determined by FEDOROV and PERFILIEVA [37] with the same apparatus.

6.412. *Scattering of neutrons produced by the D(d, n) reaction.* A very suitable homogeneous neutron source, which has been extensively used, is provided by the reaction D(*d*, *n*), which gives rise to a single neutron group. The energy set free in this exothermic reaction is 3,26 MeV, so that even with (say) 0,2 MeV deuterons from a high voltage tube, it yields homogeneous neutron beams with energies from 2 to 3 MeV according to the direction of the beam with respect to the impinging deuterons (cf. M & F, p. 52—54). Mostly, the experiments are carried out with the neutron beam parallel or perpendicular to the deuteron beam; but the apparatus used by AOKI [39] permitted a variation of the angle of the two beams and thus a controllable variation of the neutron energy in the interval 2 ... 3 MeV. With the same source, BAILEY *et al.* [46] could extend the range of neutron energies up to 6 MeV.

6.413. *Scattering of very fast neutrons by protons.* In order to obtain neutrons of still larger energies, we must at present have recourse to such (*d*, *n*) reactions as can be produced with Li, Be, B as targets; but owing to the occurrence of disintegrations into more than two particles or of excited states of the product nucleus, the energy spectra of the neutrons produced in these reactions present partly a continuous distribution, partly numerous homogeneous groups which are not easily separated. An energy selecting device has therefore to be introduced in the apparatus; no very

6.412. *Scattering of D(d,n) neutrons by protons*

Reference	Detector	Energy MeV	S 10^{-24} cm^2
BOOTH and HURST [37]	In in paraffin	3,1 *	1,7 ± 0,4**
ALLEN and HURST [40]	P	3,1 *	2,1 ± 0,2
LADENBURG and KANNER [37]	paraffin-lined I.C.; linear amplifier	2,5 *	2,11 ± 0,21
ZINN *et al.* [39]	He-filled I.C.; linear amplifier	2,85 ± 0,04	2,36 ± 0,12
KIKUCHI and AOKI [39]	I.C. with 20 atm. methane; electrometer	2,45 ± 0,05	2,28 ± 0,09
AOKI [39]	id.	2,21 ***	2,76 ± 0,10
		2,33	2,71 ± 0,08
		2,48	2,53 ± 0,10
		2,63	2,51 ± 0,08
		2,76	2,42 ± 0,10
BAILEY *et al.* [46]	argon filled I.C. with paraffin radiator	2,6	2,60 ± 0,05
		3,0	2,33 ± 0,13
		3,5	2,09 ± 0,09
		4,0	1,85 ± 0,09
		4,5	1,83 ± 0,10
		5,0	1,63 ± 0,05
		5,5	1,48 ± 0,06
		6,0	1,32 ± 0,12

* Maximum energy. — ** Value corrected according to estimate of C cross-section from data of ZINN *et al.* [39], AOKI [39] and AMALDI *et al.* [43]. — *** The absolute error may be ± 0,15 MeV, but the relative values are more accurate.

great accuracy is needed, however, since the cross-section is expected to vary slowly with energy in this region. SALANT [40] obtains a certain amount of selection by using for the detection of the neutrons the activity induced in Cu by the reactions $^{63}Cu\ (n, 2n)\ ^{62}Cu$, $^{62}Cu \rightarrow {}^{62}Ni + \beta^+$: in fact, the first reaction has a threshold at about 12 MeV, so that the Cu-detector selects from the $Li(d, n)$ continuum, for instance, the part extending from 12 to the upper end at about 15 MeV; if we make the natural assumption that the yield of the reaction increases very rapidly with increasing energy, we may locate the mean effective energy nearer the upper extremity of the interval; this effective energy can further be varied to some extent by varying the direction of the neutron beam. The same method has been used by SHERR [45] for still higher energy neutrons, produced by bombarding Li with 10,2 MeV D^+ ions: the maximum neutron energy is then 25,4 MeV and the reaction $^{12}C(n, 2n)^{11}C$ provides an energy-sensitive detector with (calculated) threshold at 20,6 MeV.

AMALDI *et al.* [43] detect the neutrons by letting them fall on a paraffin layer and counting the recoil protons with a coincidence system of three counters (fig. *6.413*); by interposing between paraffin and counters an Al

absorber of suitable thickness, they only record protons above a certain energy (which, for head-on collisions, is also the energy of the impinging

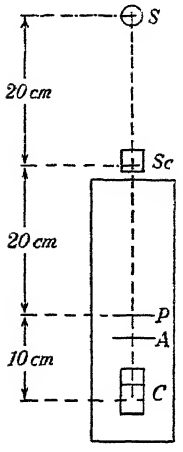

Fig. 6.413. Fast neutron scattering according to AMALDI *et al.* [43]. *S* source, *Sc* scatterer, *P* paraffin layer, *A* absorber, *C* counters.

neutrons) and in this way they contrive to select an interval of neutron energies between the lower limit defined by the absorber and the maximum energy of the spectrum. The advantage of this device is clearly the control over the lower limit of the energy interval investigated. Essentially the same method has quite recently been applied by SLEATOR [47], whose preliminary results are also quoted in the table *6.413*.

6.413. *Scattering of very fast neutrons by protons*

Reference	Source	Selector	Energy interval MeV	Mean energy MeV	S 10^{-24} cm²
SALANT and RAMSAY [40]	Li (*d*, *n*)	Cu	12 ... 15	14	0,70±0,06
				15	0,66±0,07
AMALDI *et al.* [43]	Be (*d*, *n*)	Al absorber	3,2 ... 4,6	4,1	1,73±0,06
	B (*d*, *n*)	id.	9,4 ... 13,2	12,5	0,69±0,11
	Li (*d*, *n*)	id.	11,15 ... 15	13,5	0,71±0,04
	Li (*d*, *n*)	Cu	12 ... 15	14	0,69±0,021
SHERR [45]	Li (*d*, *n*)	C	20,6 ... 25,4	25	0,39±0,04
SLEATOR [47]	Be (*d*, *n*)	Al absorber	5,68 ... 8,16	6,5	1,40±0,11
			8,24 ... 11,70	9,3	0,92±0,08
	Li (*d*, *n*)	id.	8,94 ... 13,9	10,6	0,78±0,09
			11,2 ... 15,9	12,8	0,83±0,09
			13,2 ... 18,1	14,8	0,61±0,09
			15,0 ... 19,6	16,5	0,66±0,10
			16,5 ... 21,2	18,1	0,55±0,08
			17,9 ... 22,9	19,6	0,52±0,09
			19,4 ... 24,6	21,1	0,41±0,09

6.42. *Angular distribution of proton-neutron scattering.* The investigation of the angular distribution of the scattered neutrons or recoil protons

presents considerable difficulties. The first experiments gave widely contradictory results, chiefly due, — as a critical discussion (DEE and GILBERT [37]) shows —, to the inhomogeneity of the impinging neutrons. DEE and GILBERT [37] were first able to obtain reliable results, making a very careful study of the question for $D(d,n)$ neutrons. They observed the recoil proton tracks in a large expansion chamber filled with a mixture of methane and argon at 3,5 atm. A special apparatus was designed to measure the scattering angle Θ' (*6.212–20*) from the stereoscopic pair of photographs of each track. Thanks to the high pressure, the tracks were sufficiently short to terminate in the chamber, so that the range, and consequently the energy, of each recoil proton could also be ascertained. Since for elastic scattering of neutrons of energy E (in the laboratory system), the energy E′ of the recoil protons is uniquely related to the scattering angle Θ' by the formula (see fig. *6.212*)

$$E' = E \cos^2 \Theta', \tag{1}$$

it was possible in this way to eliminate the tracks due to neutrons not coming directly from the source. The result was that, at this energy, the scattering is *isotropic in the barycentric system*; the experimental uncertainty, however, would still allow a deviation from isotropy up to about 10 %. Other experiments (KRUGER *et al.* [37], BONNER [37]) on essentially the same lines, but under less favourable conditions, led to the same conclusion. A further confirmation was given by BARSCHALL and KANNER [40], who utilized a specially constructed ionization chamber filled with a mixture of H_2 at 4 atm. and Kr at 5 atm. and derived the angular distribution of the recoil protons with the help of relation (1) from a measurement, by the linear amplifier, of their energy distribution *. Particulars of all these experiments ** with $D(d,n)$ neutrons are collected in the following table:

6.42-1. *Investigation of recoil proton tracks*

Reference	Energy of neutrons MeV	Range of scattering angle Θ'	Number of tracks
DEE and GILBERT [37]	2,44 ... 2,52	0 ... 51°	1534
KRUGER *et al.* [37]	2,44 ... 2,7	0 ... 69°	635
BONNER [37]	2,44 ... 2,47	0 ... 65°	207
BARSCHALL and KANNER [40]	2,44 ... 2,5	0 ... 55°	—

* Further experiments on the same lines have recently been announced (COON, DAVIS and BARSCHALL [46*a*]).

** Quite at variance with the measurements just mentioned, are those of the Japanese physicists (KIKUCHI *et al.* [39*b*], AOKI [39]). They compare the neutron intensity coming directly from the source with that scattered by a flat paraffin ring (the direct beam being then stopped by a block of Fe placed in the hole of the ring). From this intensity ratio they derive in a rather indirect way the value of the differential cross-section per unit solid angle $S(\Theta)$ for the mean scattering angle defined by the dimensions and position of the ring. By using rings of different sizes they get 4 values of $S(\Theta)$ in the angular

Even for much higher neutron energies, only small (if any) deviations from isotropy could be perceived. This result is well-established for energies up to about 11 MeV. Neutrons of this energy have been studied by TATEL [42]. He selects them from the continuous neutron spectrum of the Be(d, n) reaction by the same device as the Italian physicists (*6.413*), the recoil proton detector consisting of a coincidence system of two ionization chambers. He investigates their scattering by hydrogen gas in two directions, determined by the orientation of the axis joining the hydrogen cell to the detector with respect to the neutron beam (collimated by a hole in a thick wall of water). The mean scattering angles are $\Theta' = 16°$ and $\Theta' = 45°$ (i.e. $\vartheta = 148°$ and 90°, respectively); the recoil proton yields correspond, within 10 % experimental uncertainty, to isotropic distribution of the scattered neutrons.

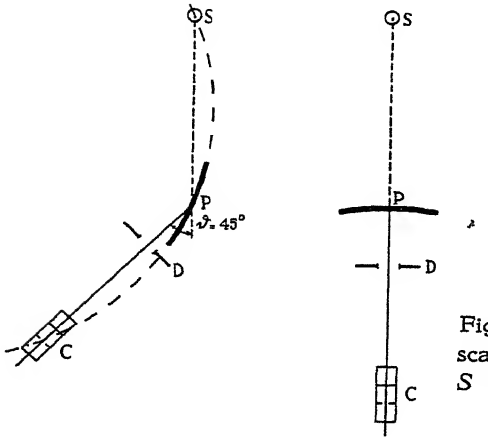

Fig. 6.42–1. Angular distribution of fast neutron scattering, according to AMALDI *et al.* [42*a*]. *S* source, *P* paraffin scatterer, *D* diaphragm, *C* counters.

In the case of neutrons of 12 ... 14 MeV, the question is not definitely settled, as different experiments lead to conflicting results. According to AMALDI *et al.* [42*a, b*], a considerable anisotropy of scattering angle distribution would occur in this energy region. The Italian physicists compare the intensity of the recoil protons from a paraffin scatterer in the direction of the incident neutron beam ($\Theta' = 0$) with that of the protons scattered at $\Theta' = 45°$ under the same geometrical conditions. For this purpose (fig. *6.42–1*), the detector (consisting of a coincidence system of three proportional counters) is rotated through an angle of 45° about the middle P of the scatterer, while the latter (which is bent into the required shape) is disposed along the circle passing through the source S, the point P and the center C of the counter system. The sources used were

range $\Theta = 23° \pm 5° \ldots 55° \pm 6°$ (thus $\Theta' = 35° \pm 6° \ldots 67° \pm 5°$) for D (d, n) neutrons of about 3 MeV. While $S(\Theta)/\cos\Theta$ turns out to be practically independent of Θ, it differs by a factor $\approx \frac{1}{2}$ from the value S/π expected in the case of S-wave scattering (S being the total cross-section). The authors attribute this discrepancy to a strong anisotropy of the angular distribution, but in view of the other results, this conclusion has very little plausibility.

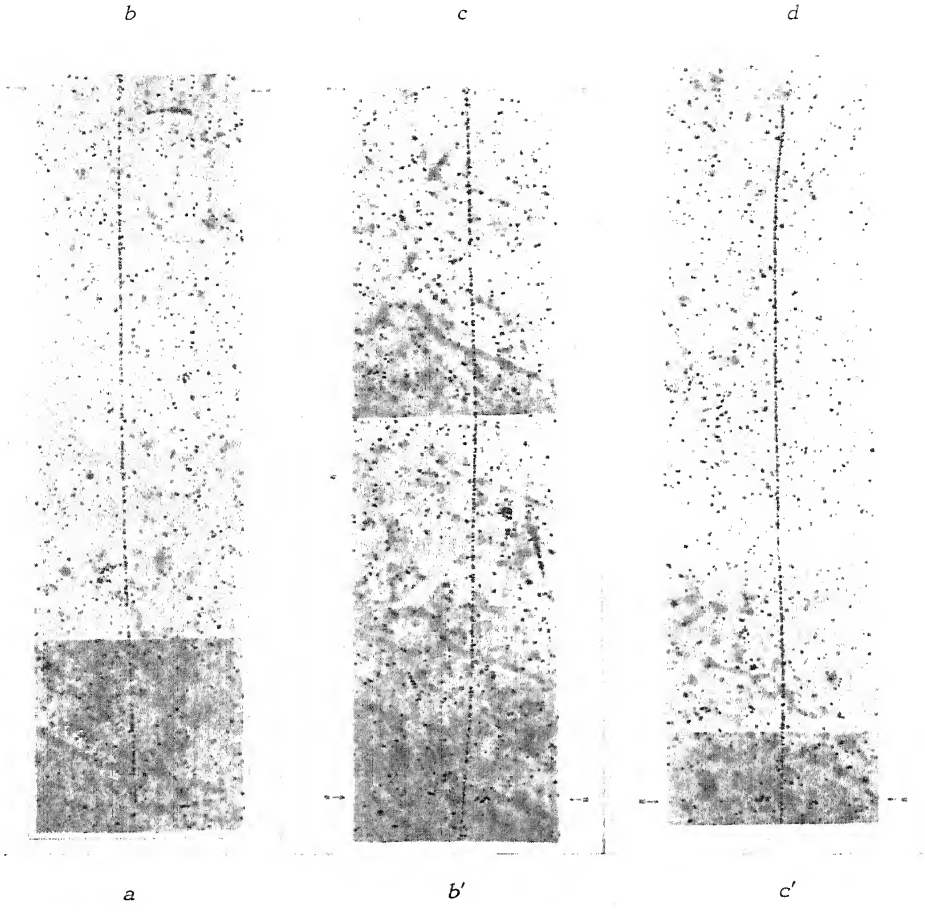

Fig. 6.42–2. Microphotograph of a 13 MeV proton track in a photographic emulsion layer (POWELL *et al.* [46*a*]). The track begins at *a* and ends at *d*; the three successive segments *a b*, *b′ c*, *c′ d* should be pieced together at the places indicated by the arrows.

the reactions B(d, n) and Li(d, n), the lower limit of the neutron spectrum being defined by Al absorbers (as above, *6.413*). The ratio $S(\pi)/S(\pi/2)$ of the values of the differential cross-section $S(\vartheta)$ per unit solid angle (in the *barycentric* system) for backward ($\vartheta = \pi$) and sideways ($\vartheta = \pi/2$) scattering, as derived from these measurements, would be:

6.42-2. *Scattering anisotropy of fast neutrons according to the Italian physicists*

Source	Energy interval MeV	Mean energy MeV	$S(\pi)/S(\pi/2)$
B (d, n)	10 ... 13,5	12,5	$0{,}71 \pm 0{,}04$
Li (d, n)	11 ... 15	13,3	$0{,}53 \pm 0{,}03$ *
	12,5 ... 15	14	$0{,}52 \pm 0{,}03$ *

* These values may be too low.

In contrast with these results, measurements performed by POWELL and his collaborators [40, 44, 46*a*, *b*] on the two fastest neutron groups from B(d, n) with mean energies 8,8 and 13 MeV, point for both groups to an angular distribution which, if it deviates at all from isotropy, would rather exhibit a small *excess* of scattered neutrons in the backward direction. Powell's method is more straightforward and much more powerful than the ingenious apparatus of Amaldi, and its results are very probably more reliable; but it is desirable that the Italian experiments be repeated with a view to tracing the cause of this puzzling discrepancy *. The method developed at Bristol consists in observing tracks of recoil protons in the emulsion layer of a photographic plate exposed to an approximately longitudinal neutron beam. The direction of the impinging neutrons can be defined with an accuracy of $\pm 0°{,}5$, and a procedure of microscopic focussing permits the measurement of the dip of a proton track with an error as small as $\pm 1°$, so that the uncertainty in the scattering angle Θ' does not exceed 2°. The measurement of the range of each track together with its scattering angle allows its assignment to a specific neutron group; special emulsions with very large grain densities have been prepared (POWELL *et al.* [46*a*]) in order to ensure an accurate definition of the tracks (fig. *6.42–2*, plate III). Account must be taken, however, of the tracks not ending in the emulsion, the thickness of which is 120 or 150 μ: for each kind of emulsion, a careful empirical determination has been made of the fraction of tracks remaining in the emulsion for different ranges and angles of dip, and before being compared with the actual results of counts of tracks, theoretical distribution curves must be corrected for this "loss" at each scattering angle (corresponding, by (1), to a different range). This circumstance practically restricts the utilization of the material

* A recent investigation by LAUGHLIN and KRUGER [47] of the angular distribution of the recoil protons produced in a high pressure cloud chamber filled with methane by neutrons of energy 12 ... 13 MeV supports the Bristol results.

to tracks with dips smaller than 10°. The method has the obvious advantage of surveying the whole angular range $\Theta' = 0 \ldots 45°$ and yielding a complete angular distribution function, which allows of a much more stringent theoretical test than the mere ratio $S(\pi)/S(\pi/2)$. The latest results on the two fast neutron groups are summarized in fig. *6.42–3, 4*; owing to

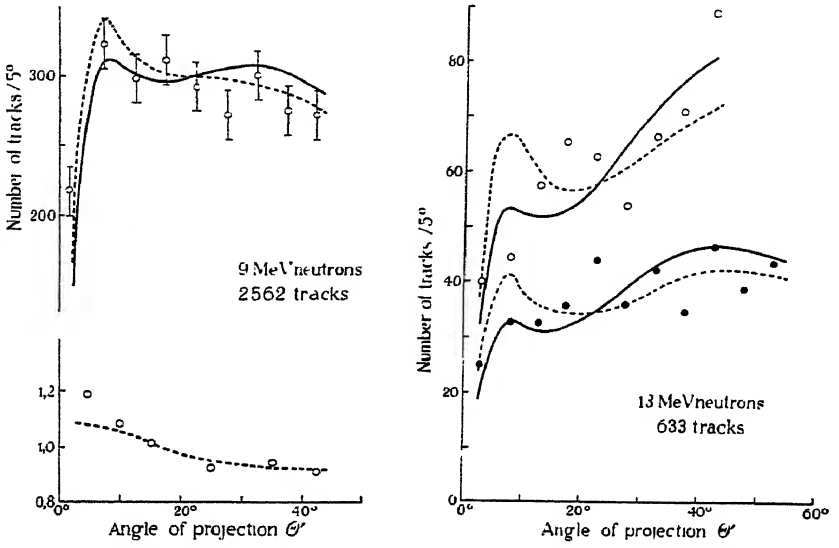

Fig. 6.42–3, 4. Angular distribution of fast neutron scattering.
. Measurements of tracks in a 150 μ emulsion layer.
o Same, combined with tracks in a 120 μ emulsion layer.
——— Isotropic distribution (corrected for "loss").
..... Distribution (corrected for "loss") according to central meson potential of symmetrical type (*8.33*) with $M_m = 225\,m$.
In the lower part of fig. *6.42–3*, the data are shown as referred to the isotropic distribution.

the statistical character of the method, an increase in accuracy may be expected from the accumulation of further data, especially on the 13 MeV group. Moreover, there is hope of applying the same analysis to neutrons of still higher energies; which would be of great importance from a theoretical point of view (*8.33*).

6.43. *Comparison with theory.* Leaving for later discussion the possible significance of the small deviations from isotropic distribution of the scattered particles indicated by Powell's experiments with 9 and 13 MeV neutrons, we may draw from the results of the preceding sub-section (*6.42*) the general conclusion that, at any rate for smaller neutron energies, we have mainly to do with S-wave scattering, in conformity with elementary theoretical considerations (*6.21*). In this energy region, we may thus compare the empirical results concerning the total scattering cross-section at various energies with the predictions of a theory of S-wave scattering, utilizing only the effective nuclear potentials for even states and adopting a definite form for these potentials. From such a detailed comparison we

may hope to derive in each case quantitative information on the ranges and strengths of the effective potentials.

6.431. *Scattering by potential well.* The calculations can be carried out in analytical form (without numerical integration of the wave-equation) only in the case of the potential well. The theory developed in *6.21*, *6.211*, *6.212*, when combined with the existence of the low energy ^1S level (*6.23*), leads to the general formula

$$\mathcal{S}(\mathsf{E}) = \frac{\pi\hbar^2}{M}\left[\frac{3\cdot {}^3f(\mathsf{E})}{\frac{1}{2}\mathsf{E}+|\varepsilon_0|} + \frac{{}^1f(\mathsf{E})}{\frac{1}{2}\mathsf{E}+|{}^1\varepsilon^{(0)}|}\right], \tag{2}$$

valid for neutron energies E satisfying the condition $\eta < 1$, i.e. (*5.11*–9; *6.212*–20) $\mathsf{E} \lesssim 10$ MeV. If the energy is expressed in MeV, we have to use the numerical value

$$\frac{\pi\hbar^2}{M} = 1{,}3\cdot 10^{-24}\ \text{cm}^2\,\text{MeV}; \tag{3}$$

the finite range corrections are given by (*6.211*–16) in terms of the potential width D (assumed to be the same for singlet and triplet states) and the scattering constants $^3a_0$, $^1a_0$.

The triplet contribution can immediately be calculated for any energy, since the parameters occurring in (2), viz. $|\varepsilon_0|$, $^3a_0$ and the G's in $^3f(\mathsf{E})$ are all determined by the deuteron binding energy $|\varepsilon_0|$ (*6.11*–1; *6.21*–14; *6.211*–19). But the analogous parameters pertaining to the ^1S level have first to be derived from the experimental data, more specifically from the zero energy cross-section (*6.23*–34; *6.211*–17)

$$\mathcal{S}(0) = 3\pi\,({}^3a)^2 + \pi\,({}^1a)^2, \tag{4}$$

$${}^3a = {}^3a_0\sqrt{{}^3f(0)} \tag{5}$$

$${}^1a = {}^1a_0\sqrt{{}^1f(0)}. \tag{6}$$

Having adopted some value for D, we begin by calculating ^3a with the help of (5), using the expression (*6.211*–16, 18) for $^3f(0)$ (with the terms in $(D/{}^3a_0)^n$ up to $n = 4$). From (4) we then obtain ^1a and solve (6) for $|{}^1a_0|$ ($^1a_0$ and ^1a are negative); it is sufficient to write (6) in the form

$$\left|\frac{{}^1a}{{}^1a_0}\right|^2 = 1 - \frac{D}{|{}^1a_0|} + 0{,}345\left(\frac{D}{|{}^1a_0|}\right)^2. \tag{7}$$

Finally $|{}^1a_0|$ yields the energy $|{}^1\varepsilon_0|$. The following table gives an idea of the variation of the various parameters with the range D:

6.431-1. *Scattering parameters in terms of width of potential well*

D 10^{-13} cm	^3a 10^{-12} cm	$-{}^1a$ 10^{-12} cm	$-{}^1a_0$ 10^{-12} cm	$^1\varepsilon(0)$ MeV	$\lvert {}^1a/{}^3a\rvert$
0	0,437	2,47 ± 0,07	2,47 ± 0,07	0,067 ± 0,004	5,66 ± 0,16
1,94	0,538	2,41 ± 0,07	2,50 ± 0,07	0,066 ± 0,004	4,48 ± 0,13
2,8	0,585	2,38 ± 0,07	2,51 ± 0,07	0,065 ± 0,004	4,07 ± 0,12

The corresponding corrections for finite range are:

6.431-2. *Finite range corrections*

D 10^{-13} cm	${}^3f(\mathrm{E})$	${}^1f(\mathrm{E})$
1,94	$1{,}51 - 0{,}02\,\mathrm{E} + 10^{-4}\,\mathrm{E}^2$	$0{,}92 - 0{,}01\,\mathrm{E} + 10^{-4}\,\mathrm{E}^2$
2,8	$1{,}79 - 0{,}04\,\mathrm{E} + 4 \cdot 10^{-4}\,\mathrm{E}^2$	$0{,}89 - 0{,}02\,\mathrm{E} + 4 \cdot 10^{-4}\,\mathrm{E}^2$
E in MeV		

Graphs of the cross-section (2) for the range values 1,94 and $2{,}8 \cdot 10^{-13}$ cm have been drawn in fig. *6.431*, in which the empirical data have also been plotted. The value $D = 1{,}94 \cdot 10^{-13}$ cm has been chosen so as to fit the cross-section $2{,}53 \cdot 10^{-24}$ cm² as measured by AOKI [39] for $\mathrm{E} \approx 2{,}5$ MeV. It will be observed that not only are AOKI's measurements in the range 2,21 ... 2,76 MeV beautifully consistent with the theoretical curve, but also those by ZINN *et al.* [39] at 2,85 MeV and by the Italian physicists (*6.413*) at 4,1 MeV fall on or very near the same curve. However, it must be noted that the energy values at which Aoki's measurements have been performed could unfortunately (owing to uncertain corrections) not be fixed with a smaller margin of error than ± 0,15 MeV; and further, — as also exhibited by the older measurements (points *B*, *K*, *L* in fig. *6.431*) —, that systematic errors tend to give too low a value of the scattering cross-section. One cannot, therefore, as shown by the figure, exclude the possibility that the true value of the range would even be as high as $2{,}8 \cdot 10^{-13}$ cm ($\approx d$): the importance of this observation will appear on a later occasion (*8.1*) *.

As regards the experimental points at higher neutron energies (12 ... 15 MeV), we cannot directly relate them to the calculated curve, since ** we ought here to take account of the waves of higher orbital momentum. The fact that also these points practically fall on the *S*-wave scattering curve indicates that *the contribution of waves of higher orbital momentum to the total scattering cross-section in the region of 12 ... 15 MeV neutron energies is very small.* This property may serve as a test for different kinds of nuclear interaction (*16.22*).

Summing up, we see that for any assumed width of the potential well, the "depths" or strengths of the effective wells for ${}^3S$ and ${}^1S$ states are determined by the binding energy of the ground state of the deuteron, and the scattering cross-section of the proton for neutrons of zero energy; the variation of the scattering cross-section with neutron energy helps to a certain

* BOHM and RICHMAN [47] show that the adjunction of a broad and shallow "tail" to the potential well tends to lower the cross-section at higher energies and thus to improve the fit with the empirical curve.

** This energy region is already outside the domain of validity of our approximate treatment of the *S*-wave scattering; but the deviation, for such energies, should not amount to more than a few percent.

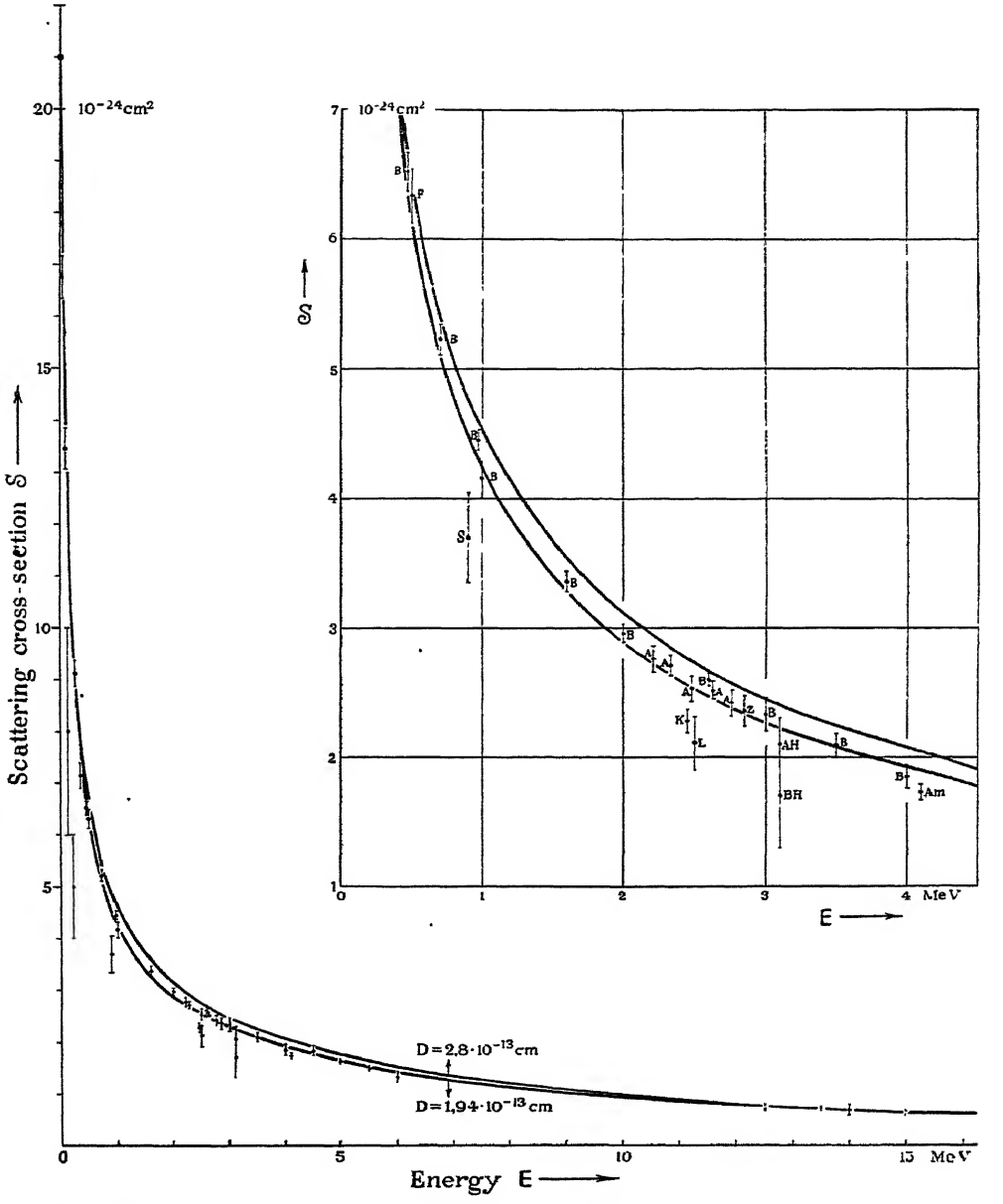

Fig. 6.431. Scattering cross-sections of protons for neutrons of various energies.
Experimental points:

A AOKI [39]
Am AMALDI *et al.* [43]
AH ALLAN and HURST [40]
B BAILEY *et al.* [46]
BH BOOTH and HURST [37]
F FRISCH [46]
K KIKUCHI and AOKI [39a]
L LADENBURG and KANNER [37]
S GOOD and SCHARFF-GOLDHABER [40]
Z ZINN *et al.* [39]

extent to fix the value of the width, but the accuracy of this determination is as yet rather poor. The explicit calculation of the potential strengths ${}^3J^{(0)}$, ${}^1J^{(0)}$ from ${}^3\varepsilon^{(0)}$, ${}^1\varepsilon^{(0)}$ is easily performed by means of formula (*5.21–5*) for an actual level and the analogous one for a virtual level; on account of the definition (*5.321–28*) adopted for such a level, one has simply to replace, in (*5.21–5*), $\sqrt{|\eta|}$ by $-\sqrt{\eta}$. The result is:

6.431-3. *Widths and depths of potential wells*

D	${}^3J^{(0)}$	${}^1J^{(0)}$
10^{-13} cm	MeV	MeV
1,94	38,1	25,4
2,8	20,9	11,8

It may be observed that, owing to the smallness of the energy of the ${}^1S$ level, the value of the strength ${}^1J^{(0)}$ of the singlet potential is quite insensitive to any experimental uncertainty affecting the scattering cross-section for zero energy neutrons. This remark is valid for any simple type of central potential.

6.432. *Scattering by other types of central potential.* In principle, the procedure followed in the case of the potential well may just as well be applied to the other simple forms of potential quoted in *5.11*, but it will be much more laborious, since for every value of the energy which is considered, the S-phase has to be computed numerically. For simplicity, we shall always assume that all effective potentials have the same range; but even so the cross-section (as exemplified by the case of the well potential) is so insensitive with respect to this range that one cannot hope to determine this parameter by a comparison with the available data. One must therefore adopt some value of the range, drawn from other evidence, such as the proton-proton scattering experiments (*7.13*). The triplet potential strength ${}^3J^{(0)}$ is then derived from ${}^3\varepsilon^{(0)}$ and the triplet contribution to the scattering cross-section for zero energy neutrons can be calculated; no great accuracy is here needed, since this contribution is small. From the empirical value of the total cross-section, the singlet contribution is thus obtained and the singlet potential strength ${}^1J^{(0)}$ necessary to account for it can finally be found. One should then check the constants determined in this way by calculating the scattering cross-section for different values of the neutron energy; but apart from the case of the well potential, treated above, this has only been done (to some extent) for the meson potential.

Let us take this last case as an illustration. From the value of $|{}^3\eta^{(0)}|$, that of ${}^3b^{(0)}$ can be calculated by formula (*5.231–25*), while ${}^1b^{(0)}$ is given in terms of ${}^1\eta^{(0)}$ by the analogous formula (*5.323–34a*). In order to determine ${}^1\varepsilon^{(0)}$, we need not compute accurately the triplet contribution to the zero energy cross-section, but (following the example of BREIT *et al.* [*39b, d*]) we may simply put it equal to the "well" value (for a width

$D = 2{,}8 \cdot 10^{-13}$ cm, say); this amounts to adopting for ${}^1\varepsilon^{(0)}$ the corresponding "well" value, given in table *6.431*–1. Inserting, in the formulae just quoted, the numerical values of ${}^3\varepsilon^{(0)}$ and ${}^1\varepsilon^{(0)}$, going over to ordinary units by means of (*5.11*–9) and introducing the meson mass $M_m = \hbar\varkappa$ instead of $\varkappa$, we get for the effective potentials

$$\begin{aligned} {}^3J^{(0)} &= 4{,}67 \cdot 10^{-4} \left(\frac{M_m}{m}\right)^2 + 5{,}57 \cdot 10^{-2} \frac{M_m}{m} - 0{,}535 \quad \text{(MeV)} \\ {}^1J^{(0)} &= 4{,}67 \cdot 10^{-4} \left(\frac{M_m}{m}\right)^2 - 0{,}965 \cdot 10^{-2} \frac{M_m}{m} \qquad \text{(MeV)}. \end{aligned} \tag{8}$$

Calculations of scattering cross-sections using these potentials will more suitably be dealt with in a later Chapter (*8.33*).

We shall here bring together in tabular form results obtained by the procedure outlined above for different types of potentials. The assumed range values are those suggested by the analysis of the proton-proton scattering data, but in the case of the meson potential figures pertaining to a whole range of values of the meson mass are quoted for future reference.

6.432. *Data on proton-neutron potentials*

Type of potential	Assumed range $\varkappa^{-1}$	${}^3J^{(0)}$	Triplet wave contribution to zero energy cross-section	${}^1J^{(0)}$	${}^1J^{(0)}/{}^3J^{(0)}$
	10^{-13} cm	MeV	10^{-24} cm^2	MeV	
Well (*5.11*-2)	2,8	20,9	3,225	11,8	0,56
Exponential (*5.11*-3)*	1,73	123,3	—	≈ 74	0,6
MORSE (*5.11*-4)**	1,4	99,06	4,47	≈ 63	$\approx 0{,}63$
Gauss (*5.11*-5)***	1,9	43,7	3,15	27,2	0,62
Meson (*5.11*-6) $M_m/m =$ 200	1,9	29,3	—	16,7	0,57
225	1,69	35,6	—	21,4	0,60
248	1,57	42,5	—	26,3	0,62
278	1,37	51,0	—	33,4	0,65
326	1,17	67,3	—	46,5	0,69
HULTHÉN (*5.11*-7)†	1	60	—	38	0,63
HYLLERAAS (*5.11*-8)††	1	112	—	77	0,69

* The value of ${}^3J^{(0)}$ is that given by RARITA and PRESENT [37] for the range indicated; that of ${}^1J^{(0)}$, which was derived by these authors from a superseded value of $\mathcal{S}$, has not been recalculated, but — following BUCKINGHAM and MASSEY [41] — the ratio q has just been put $= 0{,}6$.

** From the formula and graphs in the paper of MORSE *et al.* [36].

*** According to numerical calculations of BREIT *et al.* [39*a*]; the value of ${}^1J^{(0)}$ is obtained by interpolation from their results.

† MOSELEY and ROSEN [47].

†† EISNER and ROSEN [47]

6.5. Radiative processes

6.50. The present section is devoted to a theoretical treatment of the photo-disintegration of the deuteron and a discussion of the inverse process of radiative capture of neutrons by protons. A note is added on the theory of the disintegration of the deuteron by electron impact.

6.51. *The photo-disintegration of the deuteron.* The way in which a deuteron in its ground state of binding can be dissociated by sufficiently hard γ-radiation is apt to give valuable information on the forces holding proton and neutron together. Thus the threshold frequency of the γ-rays immediately yields the binding energy; it is in fact this method which at the moment provides us with the best value of this constant (*6.11*). But, as we shall see later (*8.34*), the angular distribution of the ejected protons also gives rise to important theoretical inferences. In order to discuss this last phenomenon, especially for relatively small frequencies, it will be necessary to take into consideration not only the disintegrations due to the electric field of the γ-ray (*photo-electric effect*) but also those produced by its magnetic field (*photo-magnetic effect*). The first depend on electric dipole transitions of the deuteron *, the second on magnetic dipole transitions. In fact, if we assume the wave-length of the γ-ray to be large compared with the dimensions of the deuteron, the interaction energy with the electromagnetic field $(\vec{E}, \vec{H})$ of the radiation (treated as constant in the spatial region of interest) may to a sufficient approximation (*4.42–5*) be written as $-\vec{E}\vec{P} - \vec{H}\vec{M}$, $\vec{P}$ and $\vec{M}$ being the electric and magnetic dipole moments.

The differential cross-section of the photo-effect for a given polarization of the incident radiation is given by

$$d\Phi = v \overline{\sum_{\tau_3', \sigma_3', \sigma_Z'} |\Psi(\infty)|^2 r^2 \, d\Omega}\,; \tag{1}$$

here v denotes the relative velocity of the ejected nucleons, determined, as well as the total kinetic energy ε, from the binding energy ε_0 of the ground state and the frequency $c\nu/2\pi$ of the incident radiation by the equation $\hbar\nu = \varepsilon + |\varepsilon_0|$; $\Psi(\infty)$ is the asymptotic value of the wave-function of the ground state, perturbed by the electromagnetic field; the bar indicates the average corresponding to the degeneracy of the ground state. The wave-function $\Psi(\infty)$ must be normalized to unity, and the magnitude of the field must be chosen so as to correspond to an intensity of one photon per cm² and per sec. Writing the interaction energy operator in the form

$$\mathcal{V}_{\text{el. magn.}} = \mathcal{W} e^{-i\nu ct} + \mathcal{W}^\dagger e^{i\nu ct}, \quad \text{with } \mathcal{W} = -\vec{E}_0 \vec{P} - \vec{H}_0 \vec{M}, \tag{2}$$

* It can be verified that electric quadrupole transitions play only a negligible role, compared to the other (PAIS [43]).

we must take

$$|\vec{E}_0|^2 = |\vec{H}_0|^2 = 4\pi \cdot \frac{\hbar\nu}{2c}. \tag{3}$$

It will appear that only the large components of the wave-function come into play; for these we may write , on account of (*4.34–28*), (*5.12–21*), (*5.111–13*),

$$\Psi(\infty) \simeq {}^3(\varrho)_1 \cdot \sum_{\sigma,l,j,m} \int d\varepsilon \cdot a_0(\varepsilon;\sigma,l,j,m;t) \cdot {}^\sigma Z_j^{(l)m} \cdot \cdot \sqrt{\frac{2c}{\pi\hbar\nu}} \cdot \frac{\sin(kr - \frac{1}{2}l\pi + {}^\sigma\delta^{(l)})}{r};$$

evaluating the coefficients a and performing the integration over ε, we find *:

$$\Psi(\infty) \simeq -{}^3(\varrho)_1 \cdot \sqrt{\frac{2\pi c}{\hbar\nu}} \cdot \frac{e^{ikr}}{r} \cdot e^{-iEct/\hbar} \sum_{\sigma,l,j,m} {}^\sigma(\tau)_0 \cdot \cdot {}^\sigma Z_j^{(l)m} \cdot \frac{e^{i\,{}^\sigma\delta^{(l)}}}{i^l} \cdot (\varepsilon;\sigma,l,j,m\,|\,\mathcal{W}\,|\,0), \tag{4}$$

where $(\varepsilon;\sigma,l,j,m\,|\,\mathcal{W}\,|\,0)$ is a matrix-element of the operator $\mathcal{W}$, the ground state being denoted by the symbol 0.

According to the considerations of *4.43*, we may take as electric dipole moment the operator given by (*4.43–11*), which clearly gives rise to transitions from the ^3S ground state to ^3P states of the continuum. As regards the magnetic moment operator, since it must operate on the ^3S state eigenfunction, it reduces to the form (*4.42–6, 7, 8, 9*); the operator $\vec{M}_{\text{nucl}}$ may be written

$$\vec{M}_{\text{nucl}} = \mu_0(\mu_n + \mu_p)\frac{\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}}{2} - \mu_0(\mu_p - \mu_n)\frac{\tau_3^{(1)} + \tau_3^{(2)}}{2}\frac{\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}}{2} - \mu_0(\mu_p - \mu_n)\frac{\tau_3^{(1)} - \tau_3^{(2)}}{2}\frac{\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}}{2}. \tag{5}$$

From this expression, it is immediately apparent that the photomagnetic transitions due to $\vec{M}_{\text{nucl}}$ lead (through the last term of (5)) to ^1S-waves. The exchange moment, as a detailed investigation (PAIS [43]) shows, likewise gives rise to transitions to ^1S states, as well as (but to a negligible extent) to ^1D states. Summarizing, we thus get a superposition of a ^3P, i.e. $\sin^2\vartheta$-distribution and a spherically symmetrical ^1S distribution of ejected nucleons (in the barycentric system).

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* The normalization must be referred to the ordinary (not reduced) energy scale, and to r instead of ξ (*5.111*).

** This method is due to BETHE, who applied it to the treatment of the atomic photo-effect. Cf. e.g. A. SOMMERFELD [39], Chapt. VI, § 7.

The total cross-section is the sum of the two effects:

$$d\Phi = d\Phi_{\text{el}} + d\Phi_{\text{magn}}. \tag{6}$$

The relevant matrix-elements are calculated without difficulty, by means of (4.32–17), from (1), (4), (2), (3); averaging further $d\Phi_{\text{el}}$ for an unpolarized incident radiation, one gets

$$\begin{aligned} d\Phi_{\text{el}} &= \frac{e^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi}{8} \, |K_{\text{el}}|^2 \sin^2\vartheta \, d\Omega \\ d\Phi_{\text{magn}} &= \frac{\mu_0^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi}{3} \, |K_{\text{magn}}|^2 \, d\Omega, \end{aligned} \tag{7}$$

where

$$\begin{aligned} K_{\text{el}} &= \int_0^\infty R_0 \cdot {}^3R_\varepsilon^{(1)} \cdot r \, dr \\ K_{\text{magn}} &= (\mu_p - \mu_n) \int_0^\infty R_0 \cdot {}^1R_\varepsilon^{(0)} \, dr - \int_0^\infty R_0 \cdot \mu(r) \cdot {}^1R_\varepsilon^{(0)} \, dr, \end{aligned} \tag{8}$$

R_0 designating the radial part of the ground state eigenfunction, and $\mu(r)$ the radial dependence of the exchange magnetic moment. The total cross-sections are given by

$$\begin{aligned} \Phi_{\text{el}} &= \frac{e^2}{\hbar} \cdot \hbar\nu \cdot \frac{\pi^2}{3} \, |K_{\text{el}}|^2 \\ \Phi_{\text{magn}} &= \frac{\mu_0^2}{\hbar} \cdot \hbar\nu \cdot \frac{4\pi^2}{3} \, |K_{\text{magn}}|^2. \end{aligned} \tag{9}$$

For frequencies just above the threshold, i.e. small velocities of the ejected nucleons, it is easy to see that the photomagnetic effect is predominant. In fact, for the eigenfunction ${}^1R_\varepsilon^{(0)}$ we may take the form (5.32–11), which for small η is proportional to

$$\chi(\xi) \cdot \frac{\sin\delta}{\eta^{\frac{1}{2}}} \cdot (\sqrt{\eta}\,\xi \operatorname{ctg}\delta + 1),$$

i.e., by (5.32–20),

$${}^1R_\varepsilon^{(0)} \sim \chi(\xi) \cdot \frac{\eta^{\frac{1}{2}}}{\sqrt{\eta} + |{}^1\eta^{(0)}|} \cdot (\sqrt{|{}^1\eta^{(0)}|}\,\xi \pm 1), \tag{10}$$

the sign of the last term being + or — according as the low energy ${}^1S$ level defining the phase is virtual or actual. On the other hand, we may in (8), replace ${}^3R_\varepsilon^{(1)}$ by the corresponding function W obtained by neglecting the nuclear potential in comparison with the centrifugal forces (5.12):

$$\begin{aligned} {}^3R_\varepsilon^{(1)} &\sim \frac{1}{\eta^{\frac{1}{2}}} \sqrt{z}\, J_{\frac{3}{2}}(z) \qquad (z \equiv \sqrt{\eta}\,\xi) \\ &\sim \frac{1}{\eta^{\frac{1}{2}}} \left(\frac{\sin z}{z} - \cos z \right) \\ &\sim \frac{1}{\eta^{\frac{1}{2}}} \cdot \frac{z^2}{3} \sim \eta^{\frac{3}{2}} \xi^2. \end{aligned} \tag{11}$$

From (8) it then follows, according to (10) and (11), that we may quite generally write, for small energies ε of the ejected nucleons,

$$\begin{aligned} |K_{\mathrm{el}}|^2 &\approx \varepsilon^{\frac{1}{2}} F_{\mathrm{el}}(\varepsilon) \\ |K_{\mathrm{magn}}|^2 &\approx \frac{\sqrt{\varepsilon}}{\varepsilon + |{}^1\varepsilon^{(0)}|} F_{\mathrm{magn}}(\varepsilon), \end{aligned} \tag{12}$$

the F's being analytic functions of ε. The formulae (9), together with (12), finally show that for small ε the photomagnetic cross-section increases with ε much more rapidly than the photo-electric one. The photomagnetic effect will chiefly affect in a sensitive way the angular distribution of the ejected nucleons (*8.34*).

To get an approximate idea of the order of magnitude of the photo-disintegration cross-sections, we may in (8) replace R_0 by its asymptotic form $e^{-\sqrt{|\eta_0|}\xi}$, with the normalization factor $\sqrt{2\varkappa\sqrt{|\eta_0|}}$. Neglecting further the magnetic exchange effect, we may then easily evaluate the K's and we get for the corresponding functions $F(\varepsilon)$ introduced in (12)

$$\begin{aligned} F_{\mathrm{el}}(\varepsilon) &\approx \frac{8}{\pi}\frac{\hbar^2}{M}\frac{\sqrt{|\varepsilon_0|}}{(\varepsilon + |\varepsilon_0|)^4} \\ F_{\mathrm{magn}}(\varepsilon) &\approx \frac{2}{\pi}(\mu_p - \mu_n)^2 \frac{\sqrt{|\varepsilon_0|}}{(\varepsilon + |\varepsilon_0|)^2}\left(\sqrt{|\varepsilon_0|} \pm \sqrt{|{}^1\varepsilon^{(0)}|}\right)^2, \end{aligned} \tag{13}$$

the $\pm$ sign pertaining to the virtual or actual character of the ${}^1S$ level. Taking account of the energy relation $\hbar\nu = \varepsilon + |\varepsilon_0|$ and of the formulae (9), (12) and (*5.32–22a*), we get for the cross-sections

$$\begin{aligned} \Phi_{\mathrm{el}} &= A\left(\frac{\hbar\nu}{|\varepsilon_0|} - 1\right)^{\frac{1}{2}}\left(\frac{\hbar\nu}{|\varepsilon_0|}\right)^{-3} \\ \Phi_{\mathrm{magn}} &= B\left(\frac{\hbar\nu}{|\varepsilon_0|} - 1\right)^{\frac{3}{2}} \frac{\left(1 \pm \sqrt{\left|\frac{{}^1\varepsilon^{(0)}}{\varepsilon_0}\right|}\right)^2}{\frac{\hbar\nu}{|\varepsilon_0|}\left(\frac{\hbar\nu}{|\varepsilon_0|} - 1 + \left|\frac{{}^1\varepsilon^{(0)}}{\varepsilon_0}\right|\right)}, \end{aligned} \tag{14}$$

$$A \equiv \frac{8\pi}{3}\frac{e^2}{\hbar}|{}^3a_0^{(0)}|^2 \quad , \quad B \equiv \frac{8\pi}{3}\frac{\mu_0^2(\mu_p - \mu_n)^2}{\hbar}.$$

Using the numerical values (*6.21–14*), (*1.21–2, 3*), (*1.22–6*), one has

$$\begin{aligned} A &= 1{,}16 \cdot 10^{-26}\ \mathrm{cm}^2 \\ B &= 1{,}52 \cdot 10^{-28}\ \mathrm{cm}^2. \end{aligned} \tag{15}$$

6.52. *Radiative capture of neutrons by protons.* When slow neutrons pass through a substance rich in hydrogen, such as paraffin or water. they are not only elastically scattered, but also strongly absorbed. This absorption is due to capture of neutrons by the protons, leading to formation of deuterons with emission of radiation. Since this process is just the inverse of the photodisintegration, its cross-section G is related to that of the latter by an equation resulting from the well-known argument of detailed

balancing. This equation, valid separately for the capture process involving an electric dipole transition as well as for that involving a magnetic dipole transition is

$$G = \frac{3}{2}\left(\frac{\hbar\nu}{p}\right)^2 \Phi; \tag{16}$$

the momentum p/c of the incident neutron is related to that of the emitted photon $\hbar\nu/c$ by the law of energy conservation, so that we may write, with (*5.111*–13),

$$G = \frac{3}{2}\frac{(\varepsilon + |\varepsilon_0|)^2}{M\varepsilon}\Phi. \tag{17}$$

The factor $\frac{3}{2}$ is the quotient of the statistical weights 2·3 of photon (two states of polarization) and deuteron (multiplicity of ground state) by those, 2·2, of free neutron and proton, arising from the spins. Combining (17) with (9) and (12), we see that for slow neutrons, the "electric" capture cross-section is negligible in comparison with the "magnetic" capture cross-section, which is inversely proportional to $\sqrt{\varepsilon}$, i.e. to the velocity v of the neutrons ("*1/v-law*"). (The reason for the smallness of the electric dipole effect is that such transitions entail an initial state of orbital angular momentum $l = 1$ and that a slow neutron with this angular momentum has too little chance to come sufficiently near the proton to be captured.)

The first measurements of the capture cross-section were performed by AMALDI and FERMI [36] according to a method developed theoretically by FERMI [36]. The mean life of a neutron of velocity v in paraffin — from which the capture cross-section immediately follows — is given by $l\,N/v$, l being the mean free path and N the mean number of free paths travelled by the neutron before capture. The first of these quantities may be measured directly by letting neutrons pass through increasing thicknesses of paraffin. The second can be deduced from a study of the diffusion of the neutrons in paraffin: the probability for a neutron to reach the surface of a paraffin block from a depth x before being captured is found to be of the form $e^{-x/l'}$, with $l' = l\sqrt{N/3}$; the determination of l' then yields N. For neutrons of ordinary thermal energies, this method gives a cross-section *

$$G = 0{,}31 \cdot 10^{-24}\ \mathrm{cm}^2. \tag{18}$$

Very nearly the same result, viz. $0{,}33 \cdot 10^{-24}$ cm², has been derived by MANLEY, HAWORTH and LUEBKE [42] from a direct measurement of the mean life-time of the neutrons in water; the accuracy of this value is estimated at about 5 %, the chief uncertainty being that in the neutron velocity.

Other determinations are obtained by comparing the capture cross-section of hydrogen with that of boron, silver or cadmium; the latter may

* Recently, however, GAMERTSFELDER and GOLDHABER [46] have given a value as low as $0{,}25 \cdot 10^{-24}$ cm², for which they claim 10 % accuracy.

in fact, on account of the smallness of scattering, be identified with the total absorption cross-section, which is readily measured. The relative measurements of the capture cross-sections are made with liquids or solutions containing the substances investigated in different concentrations; various methods of comparison can be used, a survey of which is given by VOLZ [43]. Owing to the uncertainty prevailing until recently in the absolute values of the cross-sections of the comparison substances, the precision of the absolute value of the hydrogen cross-section deduced from such relative measurements was considerably impaired. With boron as comparison element, one obtained for G values of about 0,27 to 0,31 $\cdot 10^{-24}$ cm^2 (FRISCH, HALBAN and KOCH [37], SCHULZ and GOLDHABER [45]); with cadmium (HAXEL and VOLZ [43]), one gets 0,235 to 0,32 $\cdot 10^{-24}$ cm^2, the latter determination corresponding to the most recent cadmium cross-section value (DIEBNER *et al.* [42]). A new determination, however, using monochromatized neutrons from a uranium pile and boron as comparison substance, has definitely confirmed the value (18) of the capture cross-section, which moreover can now be referred to a precise neutron velocity of 2,2 $\cdot 10^5$ cm/sec.

The cross-section (18) is much too large to be accounted for by the electric dipole effect. To the same approximation as the above formula (13), we get from (17)

$$G_{el} = C \frac{1/|\varepsilon_0|\,\varepsilon}{\varepsilon + |\varepsilon_0|}, \qquad C \equiv 4\pi \frac{e^2}{\hbar}\left(\frac{\hbar}{M}\right)^2 = 4{,}02 \cdot 10^{-29}\ \text{cm}^2, \tag{19}$$

which takes its maximum value $\frac{1}{2}C$ for $\varepsilon = |\varepsilon_0|$. It was just the need to provide for some other capture mechanism capable of explaining the large cross-section observed for slow neutrons which led FERMI [35, 36] to consider for this purpose the magnetic dipole transitions. In fact, the latter effect yields a capture cross-section following the $1/v$-law, which can thus become very large at small velocities. At this point it must be stressed that this effect exists only in so far as there is a spin dependent nuclear interaction: otherwise, there being no difference between ^1S and ^3S states, the first term in the expression (8) for K_{magn} would vanish because the eigenfunctions occurring in it would be orthogonal, while the last term would disappear because there would then be no exchange magnetic moment (*4.43*). The large capture cross-section of protons for slow neutrons therefore provides us with an additional qualitative argument of great weight for the existence of a spin dependent nuclear potential.

More quantitatively, we get *, neglecting the exchange effect as in formula (13),

$$G_{magn} \approx D \sqrt{\frac{|\varepsilon_0|}{\varepsilon}} \frac{\varepsilon + |\varepsilon_0|}{\varepsilon + |{}^1\varepsilon^{(0)}|} \cdot \left(1 \pm \sqrt{\left|\frac{{}^1\varepsilon^{(0)}}{\varepsilon_0}\right|}\right)^2,$$
$$D \equiv 4\pi \frac{\mu_0^2}{\hbar}(\mu_p - \mu_n)^2 \frac{|\varepsilon_0|}{M} = 0{,}53 \cdot 10^{-30}\ \text{cm}^2. \tag{20}$$

* Formula (20) clearly illustrates the vanishing of the magnetic effect for ${}^1\varepsilon^{(0)} = \varepsilon_0$.

Taking (table 6.431–1) $|{}^1\varepsilon^{(0)}| = 0{,}065$ MeV, i.e. $|{}^1\varepsilon^{(0)}/\varepsilon_0| = 0{,}03$, this formula yields, for thermal neutrons ($E = 0{,}026$ MeV, i.e. $\varepsilon = 0{,}013$ MeV),

$$\begin{aligned} G_{\text{magn}} &\approx 0{,}23 \cdot (1 \pm 0{,}17)^2 \cdot 10^{-24}\,\text{cm}^2 \\ &= \begin{cases} 0{,}32 \cdot 10^{-24}\,\text{cm}^2 & \text{for a virtual } {}^1S \text{ level} \\ 0{,}16 \cdot 10^{-24}\,\text{cm}^2 & \text{for an actual } {}^1S \text{ level.} \end{cases} \end{aligned} \tag{20a}$$

These results are, in the first place, of the right order of magnitude; they further distinctly favour the case of a virtual ^1S level. But (chiefly because of the neglect of the exchange effect) the calculation is not sufficiently accurate to allow us to regard this evidence as entirely conclusive.

6.53. *Disintegration of the deuteron by electron impact.* In principle, the disintegration of the deuteron can be brought about by the electromagnetic field of an impinging electron just as well as by that of a photon. Experiments to this effect have hitherto not been attempted, owing to the difficulty of producing beams of electrons of the required energies. But the analogous reaction with the ^{9}Be nucleus, which has a smaller threshold (1,63 MeV), has already been produced by GUTH and his collaborators [39a, *b*], using electrons of up to 1,75 MeV energy, accelerated by a van de Graaf generator. The theoretical treatment of this process follows the same general lines as that of the photo-effect; modifications arise on the one hand from the different form of electromagnetic field, on the other (and more important still) from the fact that the electron, unlike the photon, is not absorbed in the course of the reaction, but emerges with a diminished energy.

If the transition performed by the electron is from a state i (of momentum $\vec{p}_i^{(e)}$, energy $E_i^{(e)}$) * to a state f, so that the momentum and energy transferred to the nucleus are

$$\vec{p} = \vec{p}_i^{(e)} - \vec{p}_f^{(e)} \quad , \quad E = E_i^{(e)} - E_f^{(e)}, \tag{21}$$

the effective electromagnetic field by which the electron acts on the nucleus may, according to the principles of quantum electrodynamics **, be represented (in a notation similar to that used above in the photon case) by

$$\vec{E}_0\, e^{iEct/\hbar} + \text{complex conj.} \quad , \quad \vec{H}_0\, e^{iEct/\hbar} + \text{compl. conj.},$$

with

$$\begin{aligned} \vec{E}_0 &= \frac{1}{i}\,\frac{4\pi\hbar e}{v_i^{(e)}}\,\frac{1}{p^2 - E^2} \cdot [E\,(i\,|\,\varrho_1^{(e)}\,\vec{\sigma}^{(e)}\,|\,f) - \vec{p}\,(i\,|\,1\,|\,f)] \\ \vec{H}_0 &= \frac{1}{i}\,\frac{4\pi\hbar e}{v_i^{(e)}}\,\frac{1}{p^2 - E^2} \cdot \vec{p} \wedge (i\,|\,\varrho_1^{(e)}\,\vec{\sigma}^{(e)}\,|\,f). \end{aligned} \tag{22}$$

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* All quantities referring to the electron are distinguished by an upper index (e).

** Cf. C. MØLLER, *Ann. Physik* **14**, 531. 1932.

These formulae are only valid for sufficiently small values of the momentum transfer p; they represent in fact, to the first approximation, the field produced by the effective charge and current distribution of respective densities

$$\frac{e}{v_i^{(e)}} e^{-\frac{i}{\hbar}(\vec{p}\vec{x}^{(e)}-Ect)} (i\,|\,1\,|\,f) \quad , \quad \frac{e}{v_i^{(e)}} e^{-\frac{i}{\hbar}(\vec{p}\vec{x}^{(e)}-Ect)} (i\,|\,\varrho_1^{(e)}\vec{\sigma}^{(e)}\,|\,f).$$

The normalization factor $1/v_i^{(e)}$ ($v_i^{(e)} = c p_i^{(e)}/E_i^{(e)}$ being the initial velocity of the electron) corresponds to an intensity of one electron per cm^2 and per sec.

For a definite transition $i \to f$ of the electron, the total cross-section of the disintegration process is, exactly like that of the photo-effect (and with the same notations), the sum of an electric and a magnetic contribution

$$\begin{aligned} \Phi_{\text{el}}(i \to f) &= \frac{\pi e^2 c}{6\hbar} |K_{\text{el}}|^2 |\vec{E}_0|^2 \\ \Phi_{\text{magn}}(i \to f) &= \frac{2\pi\mu_0^2 c}{3\hbar} |K_{\text{magn}}|^2 |\vec{H}_0|^2. \end{aligned} \tag{23}$$

(Also the angular dependence of the electric and magnetic effects is the same. The differential cross-sections are obtained by multiplying the total ones by $\frac{3}{8\pi}\sin^2\vartheta\, d\Omega$ and $\frac{1}{4\pi} d\Omega$, respectively.) If the impinging electron beam is unpolarized, we have to average these quantities over the initial spin orientations; moreover, we must sum over the final spin orientations. One has, in the usual way,

$$\begin{aligned} &\tfrac{1}{2}\sum_{(\text{spin})}\sum |\vec{E}_0|^2 \\ &\quad = \tfrac{1}{2}\,\text{tr}\left[\vec{E}_0 \cdot \frac{E_i^{(e)} + \varrho_1^{(e)}\vec{\sigma}^{(e)}\vec{p}_i^{(e)} + m\varrho_3^{(e)}}{2E_i^{(e)}} \vec{E}_0 \frac{E_f^{(e)} + \varrho_1^{(e)}\vec{\sigma}^{(e)}\vec{p}_f^{(e)} + m\varrho_3^{(e)}}{2E_f^{(e)}}\right] \\ &\quad = \frac{16\pi^2\hbar^2 e^2}{v_i^{(e)2} E_i^{(e)} E_f^{(e)}}\left[-\frac{1}{4} + \frac{1}{2}\frac{E_i^{(e)2} + E_f^{(e)2}}{p^2 - E^2} - \frac{E^2 m^2}{(p^2 - E^2)^2}\right] \\ &\tfrac{1}{2}\sum_{(\text{spin})}\sum |\vec{H}_0|^2 = \frac{16\pi^2\hbar^2 e^2}{v_i^{(e)2} E_i^{(e)} E_f^{(e)}}\left[\frac{1}{4} + \frac{1}{2}\frac{p_i^{(e)2} + p_f^{(e)2}}{p^2 - E^2} - \frac{E^2 m^2}{(p^2 - E^2)^2}\right]. \end{aligned} \tag{24}$$

Finally, the cross-sections must be summed over all possible values of the momentum $\vec{p}_f^{(e)}$ of the electron after the collision. For this purpose, they have to be multiplied by the number of final states in an element of momentum space, viz.

$$\frac{p_f^{(e)} E_f^{(e)} v_i^{(e)}}{8\pi^3\hbar^3} dE_f^{(e)}\, d\Omega_f^{(e)},$$

and integrated over all angles and over $E_f^{(e)}$ from m to $E_i^{(e)} - |\varepsilon_0|$. The

angle variables occur in the integrand only through

$$p^2 - E^2 = 2\,(E_i^{(e)} E_f^{(e)} - \vec{p}_i^{(e)} \vec{p}_f^{(e)} - m^2);$$

performing the angular integration, one gets the general formulae

$$\begin{aligned}
\Phi_{\mathrm{el}}^{(e)} &= \frac{\pi}{3}\left(\frac{e^2}{\hbar}\right)^2 \int_m^{E_i^{(e)} - |\varepsilon_0|} dE_f^{(e)}\, |K_{\mathrm{el}}|^2 \cdot \\
&\quad \cdot \left[\frac{E_i^{(e)2} + E_f^{(e)2}}{p_i^{(e)2}} \log \frac{E_i^{(e)} E_f^{(e)} + p_i^{(e)} p_f^{(e)} - m^2}{(E_i^{(e)} - E_f^{(e)})\, m} - 2\frac{p_f^{(e)}}{p_i^{(e)}}\right] \\
\Phi_{\mathrm{magn}}^{(e)} &= \frac{4\pi}{3} \cdot \frac{\mu_0^2}{\hbar} \cdot \frac{e^2}{\hbar} \int_m^{E_i^{(e)} - |\varepsilon_0|} dE_f^{(e)}\, |K_{\mathrm{magn}}|^2 \cdot \\
&\quad \cdot \frac{p_i^{(e)2} + p_f^{(e)2}}{p_i^{(e)2}} \log \frac{E_i^{(e)} E_f^{(e)} + p_i^{(e)} p_f^{(e)} - m^2}{(E_i^{(e)} - E_f^{(e)})\, m}.
\end{aligned} \tag{25}$$

The energy ε of the nucleons after the disintegration, which occurs in the K's, is given by $\varepsilon = E - |\varepsilon_0|$.

For electrons of relatively low energy, we may expand the expressions (25) in powers of $W^{\frac{1}{2}}$, if

$$W \equiv E_i^{(e)} - m - |\varepsilon_0| \tag{26}$$

denotes the energy of the impinging electron above the threshold. To this end, we put

$$E_f^{(e)} = m + Wx$$

and integrate over x from 0 to 1. Since

$$\varepsilon = W(1-x)$$

is also small, we may use the expressions (12) for the K's. Expanding the integrands (but retaining in $|K_{\mathrm{magn}}|^2$ the factor $[|{}^1\varepsilon^{(0)}| + W(1-x)]^{-1}$, since $|{}^1\varepsilon^{(0)}|$ may be of the same order as W), we readily find

$$\begin{aligned}
\Phi_{\mathrm{el}}^{(e)} &\approx \frac{\pi\sqrt{2}}{6} \cdot \left(\frac{e^2}{\hbar}\right)^2 \cdot \frac{\pi}{8} F_{\mathrm{el}}(0) \cdot \frac{|\varepsilon_0|^2 + 2m^2}{|\varepsilon_0|^{\frac{1}{2}} \sqrt{(|\varepsilon_0| + 2m)\, m}} \cdot W^3 \\
\Phi_{\mathrm{magn}}^{(e)} &\approx \frac{2\pi\sqrt{2}}{3} \cdot \frac{e^2}{\hbar} \cdot \frac{\mu_0^2}{\hbar} \cdot 2\pi F_{\mathrm{magn}}(0) \cdot \frac{\sqrt{|\varepsilon_0| + 2m}}{\sqrt{|\varepsilon_0|\, m}} \cdot |{}^1\varepsilon^{(0)}| \cdot \\
&\quad \cdot \left[1 + \frac{W}{2\,|{}^1\varepsilon^{(0)}|} - \sqrt{1 + \frac{W}{|{}^1\varepsilon^{(0)}|}}\right].
\end{aligned} \tag{27}$$

Using the approximate formula (13), we get *

$$\Phi_{el}^{(e)} \approx \frac{\pi\sqrt{2}}{6} \cdot \left(\frac{e^2}{\hbar}\right)^2 \cdot \frac{\hbar^2}{M|\varepsilon_0|} \cdot \frac{|\varepsilon_0|^2 + 2m^2}{|\varepsilon_0|^4 \sqrt{(|\varepsilon_0| + 2m)m}} \cdot W^3$$

$$\approx 1{,}36 \cdot 10^{-30} W^3 \text{ cm}^2$$

$$\Phi_{magn}^{(e)} \approx \frac{2\pi\sqrt{2}}{3} \cdot \left(\frac{e^2}{\hbar}\right)^2 \cdot \frac{m}{M} \cdot \frac{\hbar^2}{M|\varepsilon_0|} \cdot (\mu_p - \mu_n)^2 \cdot \left(1 \pm \sqrt{\left|\frac{^1\varepsilon^{(0)}}{\varepsilon_0}\right|}\right)^2 \cdot \tag{28}$$

$$\cdot \frac{\sqrt{|\varepsilon_0| + 2m} \cdot |^1\varepsilon^{(0)}|}{m^2} \cdot \left[1 + \frac{W}{2|^1\varepsilon^{(0)}|} - \sqrt{1 + \frac{W}{|^1\varepsilon^{(0)}|}}\right]$$

$$\approx 1{,}6 \cdot 10^{-31} [1 + 7{,}5\, W - \sqrt{1 + 15\, W}] \text{ cm}^2,$$

W being expressed in MeV.

The formulae (28) have been derived by PETERS and RICHMAN [41]. The influence of the magnetic exchange effect has been discussed by LUBANSKI and ROSENFELD [45].

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* In the last formula, it has been assumed that the ^1S level is virtual, with $^1\varepsilon^{(0)} = 0{,}065$ MeV (table *6.431–1*).

CHAPTER VII

CENTRAL INTERACTION BETWEEN PROTONS

7.0. This Chapter will chiefly be concerned with the analysis of the scattering of protons by protons, which yields quite precise information on the effective potential of the nuclear force between two protons in a ^1S state; this effective potential depends only on the central part of the nuclear interaction (*4.431*). The influence of non-central couplings will only make itself felt for higher proton energies than those hitherto studied in sufficient detail. It has been thought advisable also to include a short account of the interesting reaction in which a collision of two protons leads to the formation of a deuteron with positon emission.

7.1. Proton-proton scattering.

7.11. *Theory.* The main force acting at large distances between two protons is the electrostatic repulsion, and it will determine the scattering as long as the colliding particles do not come into the range of the proper nuclear force. If E is the kinetic energy of the protons, the distance r_0 of closest approach in a head-on collision (angular momentum $l = 0$) is defined by $e^2/r_0 = \mathsf{E}$, whence $r_0 = (e^2/m)\,(m/\mathsf{E})$. In order that the nuclear force, of range $\approx e^2/m$, can come into play, the kinetic energy of the impinging protons must therefore be of the order of magnitude of several 100 keV. For such energies, the protons of the scatterer may be treated as free; the transition between barycentric and laboratory system of reference will be effected by the formulae (*6.212–20*). The treatment of the scattering of free protons differs from that of neutrons by protons in two respects. In the first place, the influence of the long range Coulomb interaction makes itself felt even on the asymptotic behaviour of incident and scattered wave; in the second, account must be taken of the peculiar way in which the exclusion principle modifies the scattered intensity.

Let us first consider the pure Coulomb scattering, governed by the radial equation

$$\frac{d^2R}{dr^2} + \left[k^2 - \frac{2\alpha k}{r} - \frac{l(l+1)}{r^2}\right] R = 0, \tag{1}$$

where k is again (*5.111–13*) 2π times the wave-number of the incident wave, while

$$\alpha = \frac{Me^2}{2\hbar^2 k} = \frac{Me^2}{2\hbar p} = \frac{e^2 c}{\hbar v}, \tag{2}$$

v denoting the relative velocity; indeed, with our units, v/c is the quotient of the relative momentum p by the reduced mass $\frac{1}{2}M$. For given l and k, this equation has two independent solutions of which the one is regular ($\sim r^{l+1}$), the other irregular ($\sim r^{-l}$) at the origin; the asymptotic forms of these solutions are found * to be (with corresponding normalizations)

$$\begin{aligned} L_l^{(\mathrm{reg})} &\simeq \sin(kr - l\pi/2 + \delta_c^{(l)}) \\ L_l^{(\mathrm{irreg})} &\simeq \cos(kr - l\pi/2 + \delta_c^{(l)}) \end{aligned} \tag{3}$$

with

$$\delta_c^{(l)} = \gamma^{(l)} - \alpha \log 2kr \quad , \quad \gamma^{(l)} = \arg \Gamma(i\alpha + l + 1). \tag{4}$$

They may be compared with the solutions $\sqrt{r}\, J_{\pm(l+\frac{1}{2})}(kr)$ of the differential equation without field, from which they differ asymptotically by the phase shift $\delta_c^{(l)}$. As an extension of formula (*6.21*–4), it may be seen that the regular solution

$$\Psi_c^{(+)} = \frac{1}{kr} \sum_l (2l+1)\, i^l\, P_l(\cos\vartheta)\, L_l^{(\mathrm{reg})}(r)\, e^{i\gamma^{(l)}} \tag{5}$$

is suited to describe the scattering process; in fact, its asymptotic form is **

$$\Psi_c^{(+)} \simeq e^{ikz}\, e^{i\alpha \log k(r-z)} - \frac{\alpha\, e^{ikr}}{k(r-z)}\, e^{-i\alpha \log k(r-z) + 2i\gamma^{(0)}}, \tag{6}$$

representing an incident (plane) wave and an outgoing spherical wave, both distorted by the influence of the Coulomb field; the second term we shall denote by $(\Psi_c^{(+)})_{\mathrm{scatt}}$. We have only to take care of the exclusion principle by suitably symmetrizing the solution $\Psi_c^{(+)}$. For this purpose, introduce

$$\Psi_c^{(-)}(\vartheta) \equiv \Psi_c^{(+)}(\pi - \vartheta), \quad \text{i. e.} \quad \Psi_c^{(-)}(z) \equiv \Psi_c^{(+)}(-z), \tag{7}$$

which represents, as a matter of fact, the other collision partner; if $\Psi_c^{(+)}$ describes the deflection of the impinging proton, $\Psi_c^{(-)}$ will describe the recoil of a proton from the scatterer. Recoil protons also form a part of the scattered wave, and interfere with the impinging wave in the way prescribed by the exclusion principle, according to the relative spin orientation of the two waves; for the total wave, we thus get

$$\Psi_c = {}^1J_0 \cdot {}^1(\sigma)_0\, (\Psi_c^{(+)} + \Psi_c^{(-)}) + \sum_{m_s} {}^3J_{m_s} \cdot {}^3(\sigma)_{m_s}\, (\Psi_c^{(+)} - \Psi_c^{(-)}), \tag{8}$$

with arbitrary amplitudes ${}^\sigma J_{m_s}$. Owing to the symmetry properties of the

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* Cf., e.g., SOMMERFELD [39], Chapt. II, § 7. In the notation of W & W *16.1*, the regular solution is $\sim M_{i\alpha, l+\frac{1}{2}}(2ikr)$.

** Cf. SOMMERFELD [39], Chapt. II, § 9. For a deduction of (6) from (5), see W. GORDON, *Z. Physik* **48**, 180. 1928.

Legendre polynomials, $\Psi_c^{(+)} \pm \Psi_c^{(-)}$ is simply twice an expression of the form (5), in which the summation over l must be restricted to the even or odd values of l, respectively; this is, of course, precisely the restriction embodied in (*4.331–23*) when the isotopic eigenfunction is ${}^3(\tau)_{-1}$. Asymptotically, we have, by (6),

$$(\Psi_c^{(+)} \pm \Psi_c^{(-)})_{\text{scatt}} \simeq -\frac{\alpha e^{ikr}}{2kr} e^{-i\alpha \log 2kr + 2i\gamma^{(0)}} \left(\frac{e^{-i\alpha \log \sin^2 \vartheta/2}}{\sin^2 \vartheta/2} \pm \frac{e^{-i\alpha \log \cos^2 \vartheta/2}}{\cos^2 \vartheta/2} \right). \quad (9)$$

From (8) we obtain for the differential scattering cross-section

$$d\mathcal{S}_c = r^2 \, d\Omega \, [\, |{}^1J_0|^2 \cdot |\, \Psi_c^{(+)} + \Psi_c^{(-)}|^2_{\text{scatt}} + \sum_{m_s} |{}^3J_{m_s}|^2 \cdot |\, \Psi_c^{(+)} - \Psi_c^{(-)}|^2_{\text{scatt}}],$$

the incident wave being normalized not to unity, but * to 2; for an unpolarized wave, we must therefore make all $|{}^\sigma J_{m_s}|^2 = \frac{1}{4}$ and get from (9)

$$d\mathcal{S}_c = d\Omega \frac{\alpha^2}{k^2} \left[\frac{1}{(2\sin^2 \vartheta/2)^2} + \frac{1}{(2\cos^2 \vartheta/2)^2} + \frac{(\frac{1}{4} - \frac{3}{4}) 2 \cos(\alpha \log \operatorname{tg}^2 \vartheta/2)}{4 \sin^2 \vartheta/2 \cos^2 \vartheta/2} \right],$$

or, going over to the laboratory system,

$$d\mathcal{S}_c = 4 \cos \Theta \, d\Omega_{\text{lab}} \frac{\alpha^2}{4k^2} \left[\frac{1}{\sin^4 \Theta} + \frac{1}{\cos^4 \Theta} - \frac{\cos(\alpha \log \operatorname{tg}^2 \Theta)}{\sin^2 \Theta \cos^2 \Theta} \right]. \quad (10)$$

This is the well-known formula first derived by Mott. The first term is just Rutherford's classical expression; the second is the same expression for the recoil protons; the third results from the exclusion principle.

The modification of the Coulomb scattering, due to a short range potential, can now be derived in a way entirely analogous to the method of *6.21*. For given l and k, the asymptotic form of the solution may be written in the form

$$L_l \simeq \sin(kr - l\pi/2 + \delta_c^{(l)} + {}^\sigma\delta^{(l)}), \quad (11)$$

as a linear combination of the two asymptotic solutions (3); the constant phase-shift ${}^\sigma\delta^{(l)}$ depends mainly on the short range potential ${}^\sigma V^{(l)}$ (and to a lesser degree also on the presence of the Coulomb potential). The general solution

$$\Psi = \Sigma' {}^\sigma\mathfrak{A}_j^{(l)m} \cdot {}^\sigma Z_j^{(l)m} \cdot \frac{1}{r} L_l(r),$$

in which Σ' denotes the restricted summation over even or odd l according as $\sigma = 1$ or 3, must be determined in such a way that $\Psi - \Psi_c$ reduce asymptotically to an outgoing wave only. This condition, using (8) and

* Cf. N. MOTT and H. MASSEY, *The theory of atomic collisions* (1933), p. 69 (footnote).

(5), gives for the scattered wave

$$\Psi_{\text{scatt}} = \sum_{\sigma, m_s} {}^{\sigma}J_{m_s} \cdot {}^{\sigma}(\sigma)_{m_s} \cdot {}^{\sigma}\Psi_{\text{scatt}}, \tag{12}$$

with

$$\begin{aligned} {}^{\sigma}\Psi_{\text{scatt}} \simeq (\Psi_c^{(+)} \pm \Psi_c^{(-)})_{\text{scatt}} \\ + \Sigma' \sqrt{4\pi(2l+1)}\, Y_l^0 \frac{e^{ikr}}{ikr} e^{i(2\gamma^{(l)} - \alpha \log 2kr)} [e^{2i\,{}^{\sigma}\delta^{(l)}} - 1], \end{aligned} \tag{13}$$

the double sign being + or — according as $\sigma = 1$ or 3; the symbol Σ' has just been explained. By (9) we may write

$$\begin{aligned} {}^{\sigma}\Psi_{\text{scatt}} \simeq \frac{e^{ikr}}{ikr} e^{-i\alpha \log 2kr} \left[\frac{\alpha}{2i} \left(\frac{e^{-i\alpha \log \sin^2 \vartheta/2}}{\sin^2 \vartheta/2} \pm \frac{e^{-i\alpha \log \cos^2 \vartheta/2}}{\cos^2 \vartheta/2} \right) e^{2i\gamma^{(0)}} \right. \\ \left. + \Sigma' \sqrt{4\pi(2l+1)}\, Y_l^0\, e^{2i\gamma^{(l)}} (e^{2i\,{}^{\sigma}\delta^{(l)}} - 1) \right]. \end{aligned} \tag{14}$$

For unpolarized waves, the cross-section is built up from the singlet and triplet contributions in the ratio (1 : 3):

$$\begin{aligned} d\mathcal{S} &= \tfrac{1}{4}\, d^1\mathcal{S} + \tfrac{3}{4}\, d^3\mathcal{S} \\ d^{\sigma}\mathcal{S} &= |{}^{\sigma}\Psi_{\text{scatt}}|^2\, r^2\, d\Omega. \end{aligned} \tag{15}$$

Formula (14) shows that the total scattering consists of the pure Coulomb scattering, as given by Mott's formula (10), and of an "anomaly" due to the scattering of the waves of different angular momenta by the nuclear potential and to interference between these waves and the Coulomb wave. Keeping in mind that the phases ${}^{\sigma}\delta^{(l)}$ decrease with increasing l, we may from (14), (15) evaluate the cross-section with increasing accuracy by successively taking into account the terms corresponding to $l = 0, 1, 2, \ldots$. The main contribution will be the S-wave anomaly, coming from the singlet configuration (antiparallel spins of the colliding protons); it arises from the nuclear scattering of the S-wave and from its interference with the Coulomb wave:

$$\begin{aligned} d\mathcal{S}^{(0)}_{\text{anom}} = 4 \cos\Theta\, d\Omega_{\text{lab}} \frac{\alpha^2}{4k^2} \left[\frac{4}{\alpha^2} \sin^2 \delta \right. \\ \left. - \frac{2}{\alpha} \sin\delta \left(\frac{\cos(\delta + \alpha \log \sin^2\Theta)}{\sin^2\Theta} + \frac{\cos(\delta + \alpha \log \cos^2\Theta)}{\cos^2\Theta} \right) \right], \end{aligned} \tag{16}$$

where we have used the laboratory variables and put, for simplicity, $\delta \equiv {}^{1}\delta^{(0)}$. Next in importance comes a P-wave anomaly, from the triplet configuration (parallel spins); this anomaly also comprises a pure scattering and an interference term. There follows a D-wave anomaly, again from the singlet configuration, which consists of three terms, since there is also interference between the D- and the S-wave. These anomalies have been calculated by BREIT and his collaborators [36a, 39a], who also give an

elaborate discussion of the procedure best suited to derive from the observed angular distribution of the scattering intensity the values of the phases at the corresponding energy. It appears that for most of the hitherto utilized proton energies (in fact for energies up to a few MeV) practically the whole observed deviation from Mott's formula is due to the S-wave anomaly.

About the behaviour of the ratio

$$\Re \equiv \frac{d\mathcal{S}_{\text{obs}}}{d\mathcal{S}_c} \approx 1 + \frac{d\mathcal{S}^{(0)}_{\text{anom}}}{d\mathcal{S}_c}, \tag{17}$$

some theoretical predictions can be made, which, as we shall see, allow us to draw important conclusions on the nature of the nuclear forces. To begin with, we expect, on account of (10) and (16), the ratio $\Re$ at a given energy to exhibit a maximum or minimum for $\Theta = 45°$, about which value it will vary symmetrically with angle, tending to unity for $\Theta \to 0$ and $\Theta \to 90°$. But the nature of the extreme, as well as the more detailed behaviour of the angular variation will depend in a characteristic way on the *sign* of the nuclear proton-proton force. If this force is repulsive, it will always enhance the effect of the Coulomb field, so that $\Re$ will always be larger than unity and have a maximum at $\Theta = 45°$. The case of an attractive force, however, is more complicated. If the energy of the impinging protons is sufficiently high, the nuclear field (which now counteracts the Coulomb repulsion) will predominate, giving $\Re > 1$; but this preponderance will depend on the scattering angle, since the Coulomb scattering becomes very large at small angles. As the energy decreases, and the influence of the nuclear potential consequently becomes weaker, $\Re$ will therefore become < 1 at small angles, while remaining > 1 at larger angles, and the region where it is < 1 will progressively expand. At still lower energies, $\Re$ will have become < 1 for all angles; by further decrease of the energy, the maximum of $\Re$ at 45° will change over to a minimum. During all this variation, the value of $\Re$ at any angle will decrease until it reaches a minimum; after which, as the Coulomb field begins to predominate over the nuclear field (at that angle), $\Re$ increases again towards the limiting value unity. We thus see that, in contrast to the case of proton-neutron scattering (the S-part of which only depends on the absolute values of the phases), the scattering of protons by protons is quite sensitive to the sign of the nuclear potential; this is due, of course, to the fact that we have here to do with an interference effect between Coulomb and nuclear scattering.

7.12. *Analysis of experiments.* The technique of absolute measurements of proton-proton scattering has been developed to a high degree of accuracy, chiefly by TUVE, HEYDENBURG and HAFSTAD [36, 38, 39] and by HERB and his collaborators (HERB *et al.* [39], RAGAN *et al.* [41]). The apparatus is clearly described by fig. 7.12–1, which represents TUVE

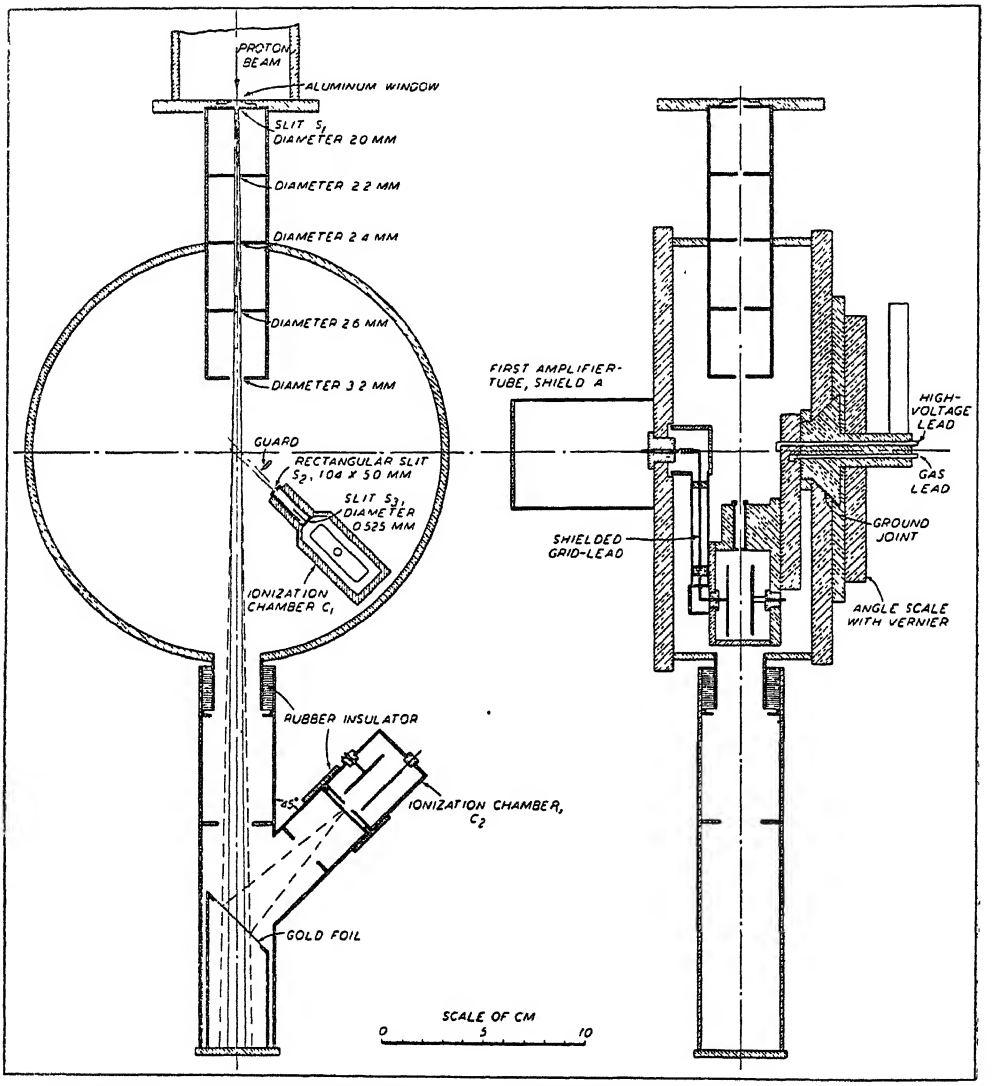

Fig. 7.12–1. Proton-proton scattering apparatus, according to TUVE, HEYDENBERG and HAFSTAD [39].

et al.'s improved design. The scattering chamber, in the form of a flat cylinder, is filled with hydrogen at 12 mm pressure; it contains a movable detector which registers the protons scattered at various angles in a volume defined by the divergence of the primary (incident) proton beam and the convergence of the detector slit-system. As a detector, the ionization chamber connected with a linear amplifier ceases to be satisfactory for protons with a residual range less than about 3 mm; to record slower protons (TUVE *et al.* [38], RAGAN *et al.* [41]), one has to replace it by some variety of point counter.

For each value of the energy of the incident protons, the results of the experiments are conveniently expressed by giving the ratio $\mathfrak{R}$, defined by

formula (17), of the observed differential cross-section to that corresponding to pure Coulomb-scattering, in terms of the scattering angle Θ. Since $\mathfrak{R}(\Theta)$ is expected to be symmetrical about $\Theta = 45°$, most of the measurements are limited to the angular range 0° ... 45°, but some of them (TUVE *et al.* [38, 39], RAGAN *et al.* [41]) have been extended up to 60° to check the symmetry. Results obtained by this method are now available for the whole energy range between 0,175 and 2,4 MeV. Besides, an investigation of the scattering of 4,2 MeV protons by protons has been carried out by MAY and POWELL (POWELL [47*a*]), using the latter's photographic method (*6.42*). All these measurements will be referred to as the "first group".

A "second group" of experiments, performed by WILSON and his collaborators [47*a*, *b*, *c*], covers the range of 8 ... 14,5 MeV proton energies. The protons from a cyclotron are scattered by the hydrogen atoms in a cellophane or Nylon foil. The scattered and the recoiling particles are recorded in coincidence by two counters fixed on rotating arms at 90°, with the scattering foil at the centre of rotation; in this way, spurious particles are eliminated. In this energy region, there is no point in considering the ratio $\mathfrak{R}$: the results of the (relative) measurements are described by giving, for each energy (E = 8, 10 and 14,5 MeV), the ratio $\mathfrak{S}(\Theta)$ of the differential cross-section at angle Θ to that at $\Theta = 45°$ (cf. fig. *7.131*). For E = 8 MeV, the absolute scattering cross-section per unit solid angle at $\Theta = 45°$ has also been determined and found to be $(1{,}7 \pm 0{,}1) \cdot 10^{-25}$ cm^2.

Already the first experiments by TUVE *et al.* [36] (fig. *7.12–2*) allowed

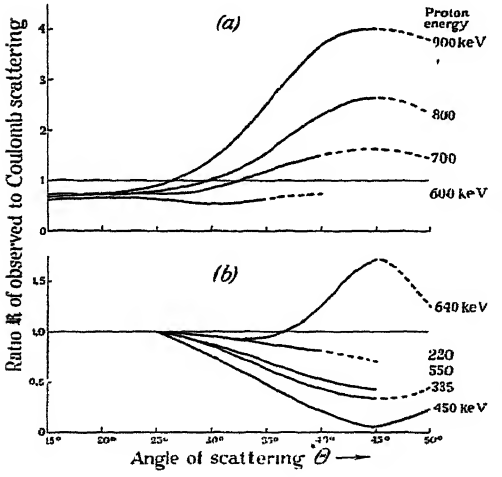

Fig. 7.12–2. Proton-proton scattering. Ratio $\mathfrak{R}$ of observed to pure Coulomb scattering as a function of scattering angle Θ. Experiments by TUVE, HEYDENBURG and HAFSTAD:
(*a*) according TUVE *et al.* [36],
(*b*) according TUVE *et al.* [37].
Relative measurements; $\mathfrak{R}$ is adjusted to unity at $\Theta = 25°$.

BREIT, CONDON and PRESENT [36*a*] to conclude (upon the natural assumption that *S*-wave scattering is predominant) that *the nuclear proton-proton potential is attractive*: this follows indeed, as explained in *7.11*, from the occurrence of $\mathfrak{R}$-values smaller than unity at small angles, the angular region in question spreading towards larger angles as the proton energy decreases. This qualitative inference was further strikingly confirmed by new experiments (TUVE *et al.* [38]) extending to lower

energies: it was observed (fig. 7.12–2) how the ratio $\Re(45°)$ changed from a maximum to a minimum. For an energy of 0,45 MeV, the scattering intensity at 45° almost vanished, owing to destructive interference between Coulomb and nuclear field. At still lower energies, the ratio $\Re(45°)$ began to increase again, quite in conformity with theoretical expectation.

The subsequent experiments of the first group (HERB *et al.* [39], TUVE *et al.* [39], RAGAN *et al.* [41]) aimed at increasing the precision and extending the range of proton energies explored. The mutual consistency of the results could be checked in overlapping energy regions (see table 7.12 and fig. 7.12–3), and was found to be very satisfactory. The empirical

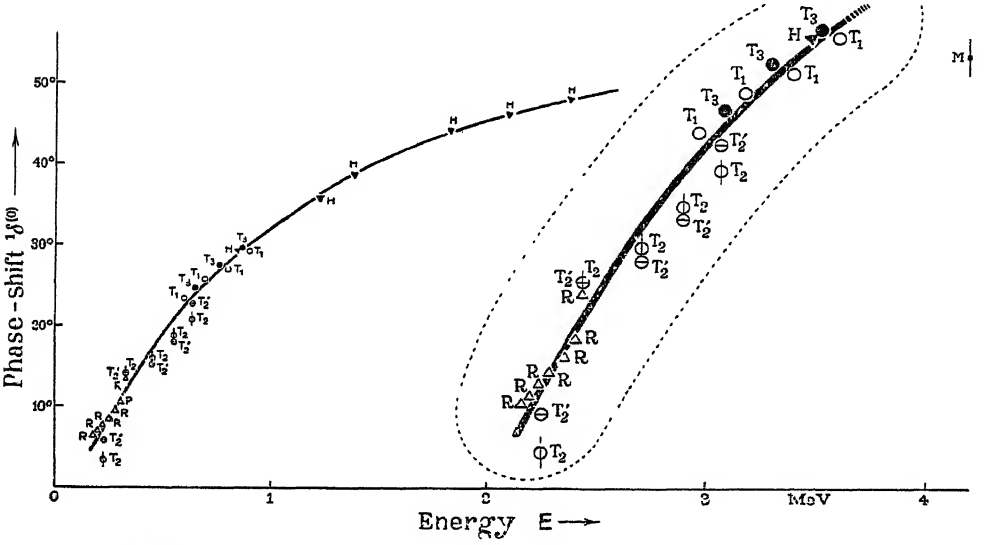

Fig. 7.12–3. Proton-proton scattering. S-phase derived from experiments:

R RAGAN *et al.* [41]	T_1 TUVE *et al.* [36]
H HERB *et al.* [39]	T_2 TUVE *et al.* [38]
M MAY and POWELL (POWELL [47a])	T_2' the same, using only the scattering at $\Theta = 45°$
	T_3 TUVE *et al.* [39]

data were carefully analysed by BREIT and his collaborators (BREIT [36*a*, 39*a*], CREUTZ [39]) in order to derive the phases $\delta^{(l)}$ as functions of the proton energy. The S-wave scattering — in the first instance assumed to be responsible for the whole effect — is in fact found to account for it with deviations of at most a few percent (in many cases less than 1 %). These deviations cannot consistently be attributed to P- or D-wave scattering, which leads to the conclusion that the latter effects, in the energy range under consideration, are too small to be disclosed even by such accurate measurements.

From (10), (16), (17) it follows that

$$\Re(45°) = 1 - \frac{2}{a} \sin\delta \cos(\delta - a \log 2) + \frac{1}{a^2} \sin^2\delta, \qquad (18)$$

and from this formula, with $\alpha^{-1} \approx 5 \ldots 10$, two essentially different values of the S-phase δ may in general be deduced, one of them lying in the first quadrant, the other in the fourth. The latter would correspond to a repulsive potential. Quite apart from the behaviour of $\mathfrak{R}$ at smaller angles, it can be excluded, however, by application of a useful criterion due to Wheeler (cf. BREIT *et al.* [36a]): If the radial wave-function R keeps the same sign in the interval $0 \ldots r'$, the ratio $\left(\frac{1}{R}\frac{dR}{dr}\right)_{r=r'}$ decreases with increasing energy. [The proof results immediately from the first formula in the footnote to (*5.12*–21).] The condition for R is fulfilled in the lowest (nodeless) S state up to the distance r' at which the function is sufficiently well represented by its asymptotic form, so that $\left(\frac{1}{R}\frac{dR}{dr}\right)_{r=r'}$ can be calculated in terms of δ. It is then seen that the fourth quadrant value contradicts Wheeler's criterion. This gives additional support to the above deduction that the proton-proton potential is attractive.

The results of the analysis of the first group of experiments are summarized in the following table and represented on fig. *7.12*–3.

7.12. S-phases derived from first group of proton-proton scattering experiments

E keV	$^1\delta^{(0)}$ degrees	Estimated error on phase degrees	Reference
176,5	6,3		
200,2	6,7		
225,9	7,4		
249,5	8,5	0,5	RAGAN *et al.* [41]
275,3	9,5		
298,3	10,5		
321,4	13,4		
220	3,5	0,8	
325	14,1	—	
450	16,0	0,4	TUVE *et al.* [38]
550	18,7	0,6	
640	20,8	0,7	
670	24,80		
776	27,42	0,4	TUVE *et al.* [39]
867	29,41		
600	23,5	1,6	
700	25,87	0,71	TUVE *et al.* [36]
800	27,13	0,44	
900	29,55	0,25	
860	29,28		
1200	35,94		
1390	38,96	0,4	HERB *et al.* [39]
1830	44,02		
2105	46,18		
2392	48,08		
4200	54,0	2,5	MAY and POWELL (POWELL [47a])

The experiments of the second group have not been analyzed in the same detailed way, but the observed differential cross-sections have been

directly compared with the theoretical ones; we shall come back to this point in *7.131* below.

7.13. *Derivation of nuclear potential parameters from the observed phases.* There is unfortunately no analytical method (at least in quantum theory *) to derive the expression for the scattering potential from the known values of the phases at various energies. The only possibility is to start from some assumed type of potential, involving undetermined parameters, and to try to adjust the values of these parameters in such a way that the calculated phases agree with observation. This is a very unsatisfactory procedure, since — besides involving tedious numerical integrations of the wave-equation — it can at best only yield several possible types of nuclear potential, without any means (on internal evidence) of removing the resulting ambiguity. An encouraging feature, however, is that these various solutions all lead, as we shall see, to practically equivalent general conclusions.

For a detailed account of the special procedures required for the calculation of the phases corresponding to a given form of nuclear potential, the reader is referred to the papers already cited of BREIT and his collaborators [*36a*, *39a*] **. In the latter paper [*39a*], refined methods of series expansions have been elaborated in order to check the results of numerical integrations. We shall just mention here that phases of higher angular momentum can be estimated by a formula

$$\delta^{(l)} \approx -\frac{M}{\hbar^2 k}\int_0^\infty V^{(l)}(r)\,[L_l^{(\mathrm{reg})}(r)]^2\,dr, \tag{19}$$

entirely analogous to (*5.33–37*) and derivable by the same argument, starting from an appropriate identity, similar to (*5.121–25*). A slight modification of the argument permits an estimation of the change $d\delta$ of the phase-shift δ resulting from a small change dV of the nuclear potential ***; if R is the radial wave-function for the original choice of the potential, we have to a first approximation

$$d\delta \approx -\frac{M}{\hbar^2 k}\int_0^\infty dV(r)\cdot R^2\,dr. \tag{20}$$

More accurate formulae have been derived by BREIT *et al.* [*36a*, *39a*, *39d*], and used in particular [*39d*] to compare the effects due to potentials of different shapes, — the difference arising from a change either of the parameter values or of the potential type itself.

Let us now survey the results which have been obtained with various

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* On the analogous classical problem, see HOYT and FRYE [40]. In this paper, the semi-classical approximation based on an expansion of the phase of the wave-function in powers of $\hbar$ is also treated; but it is unsuited for nuclear problems.

** LANDAU and SMORODINSKY [44] have obtained by a semi-empirical method a general formula roughly representing the variation of the phases with energy.

*** This method is due to Mott; it has first been worked out by H. TAYLOR, *Proc. R. S.* **A 136**, 605. 1932.

simple types of potentials; the most extensively studied are the well, Gauss and meson potentials. For the first two types, calculations of S-wave scattering and estimates of P-wave phases have been extended up to 10 MeV by THAXTON and HOISINGTON [39*b*]. In all cases, the best values of the range constant $\varkappa$ and the strength ${}^1J_1^{(0)}$ have been determined so as to fit the measurements of the first group at higher proton energies (HERB *et al.* [39], TUVE *et al.* [39]); later on (THAXTON [41]), the test was extended to the low energy results (RAGAN *et al.* [41]). As to May and Powell's results for 4,2 MeV protons, they agree in a general way with the theoretical predictions, but are not accurate enough for the purpose of discriminating between different potentials.

Potential well. In this case, the high energy values of the phases can be fitted to two potentials of different widths (and depths); but the low energy measurements just quoted are useful in removing this ambiguity: they definitely establish that the best fit at all energies is obtained with a well-width very nearly equal to the electron radius d. The corresponding depth is found * to be 11,35 MeV (BREIT *et al.* [36*a*, 39*a*, 39*d*]).

Gauss potential. Also for this type of potential, BREIT *et al.* [39*a*, *d*] were able to determine uniquely the constants fitting all experimental results, thanks to the remarkable accuracy of these results. From calculations of the S-wave scattering, they found $\varkappa^{-1} = 1{,}9 \cdot 10^{-13}$ cm, $J = 26{,}3$ MeV. But the small P-wave anomaly has also been calculated by THAXTON and MONROE [39*a*] and shown not to be significant.

Meson potential. In contrast to the types just discussed, the meson potential does not allow of unique determination by comparison with the observed phases. BREIT *et al.* [39*d*] showed that no sufficiently close fit with these phases could be achieved with a meson mass of 200 m, and they concluded that the best possible fit required a meson mass as high as 326 m. However, in a renewed investigation, LUBANSKI and DE JAGER find

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* It is convenient first to calculate the scattering upon the assumption that the Coulomb potential does not act inside the well; the depth so determined, viz. 10,5 MeV, has then to be increased, in order to compensate the effect of the Coulomb repulsion. The correction (of the order of magnitude $e^2/D \approx e^2/d = m$) can easily be estimated with any desired accuracy (BREIT *et al.* [39*a*]). A first approximation, for instance, is obtained from the expression (20) for the change of phase resulting from an assumed change of potential: for this change of potential, inside the well, we take the sum of e^2/r and a constant $-dJ$, and express that the resulting total change of phase is zero; this yields for the increase of depth needed

$$dJ = \frac{e^2}{D} \cdot \frac{a \int_0^{\alpha} \frac{\sin^2 u}{u}\, du}{\int_0^{\alpha} \sin^2 u\, du} = \frac{e^2}{D} \cdot \frac{a\left(\log 2a + 0{,}5772 + \int_{2a}^{\infty} \frac{\cos u}{u}\, du\right)}{a - \sin a \cos a},$$

with

$$a \equiv \frac{\sqrt{M(J + \frac{1}{2}E)}}{\hbar} \cdot D.$$

just as close a fit with $M_m = 278\,m$, whereas even with a mass value $M_m = 248\,m$, the deviations of the theoretical figures from the empirical ones do not exceed four times the estimated errors of measurement. Still, this would seem to point to a range of nuclear force rather smaller than that corresponding to the mass of the cosmic ray mesons observed at sea level. But although the available experimental results exhibit a very satisfactory internal consistency, they might be affected by small systematic errors, so that it is difficult to judge whether this discrepancy is real or spurious. To decide this issue by new proton-proton scattering experiments is one of the most important tasks in the field of research bearing on nuclear forces.

For later reference, all data thus far obtained on proton-proton potentials are here collected in tabular form:

7.13. *Parameters of proton-proton potentials from scattering data*

Type of potential	Reference	Best parameter values		Observations
		Range $\varkappa^{-1}$ 10^{-13} cm	Strength J MeV	
Well (*5.11-2*)	BREIT *et al* [36a, 39a, *d*]	2,8	11,35	
Exponential(*5.11-3*)	BREIT *et al.* [39*d*]	1,73	70,3	
MORSE (*5.11-4*)	THAXTON and MONROE[40a]	1,4	61,0	
Gauss (*5.11-5*)	BREIT *et al.* [39a, *d*]	1,9	26,3	
Meson (*5.11-6*) $M_m/m =$ 200	BREIT *et al.* [39*d*]	1,9	17,45	Poor fit with data
248	LUBANSKI and DE JAGER	1,55	25,8	No close fit with data
278	id.	1,37	32,6	Close fit with data
326	BREIT *et al.* [39*d*]	1,17	45,8	Close fit with data
$w(\xi) = \xi^{-2}$	id.	—	—	Unsuited

7.131. *Comparison of second group of scattering data with theory.* The second group of scattering data, relating to proton energies in the range 8 ... 14,5 MeV, might be expected to give evidence of the incipient influence of P-scattering and accordingly to yield information on the effective potential in the ^3P-configuration of the two protons. To this end, the empirical curves giving $\mathfrak{S}(\Theta)$ (*7.12*) have been compared with the values calculated on the assumption that the effective potential in the ^3P configuration is a well of the same width as the ^1S one. The calculation is carried out both for an attractive and for a repulsive ^3P potential, while the ^1S potential is taken to be attractive, in conformity with the foregoing analysis. Unfortunately, the results of such a comparison, as exemplified by fig. *7.131*, do not as yet allow us to draw any far-reaching conclusions. If anything, they are in favour of a *repulsive* ^3P*-potential.*

Nevertheless, PEIERLS and PRESTON [47], attempting a more detailed analysis of the 10 MeV results, show that these results would indicate a $1S$ phase of 52,5° and a ^3P phase of about $-0{,}8°$. Assuming well potentials of somewhat smaller width, viz. $D = 2{,}5 \cdot 10^{-13}$ cm, they find that the ^3P phase can be accounted for by a repulsive potential of about 10 ± 3 MeV;

this figure may be compared (*11.21*) with the depth ${}^3J = 25$ MeV of the ${}^3S$ potential for the same width.

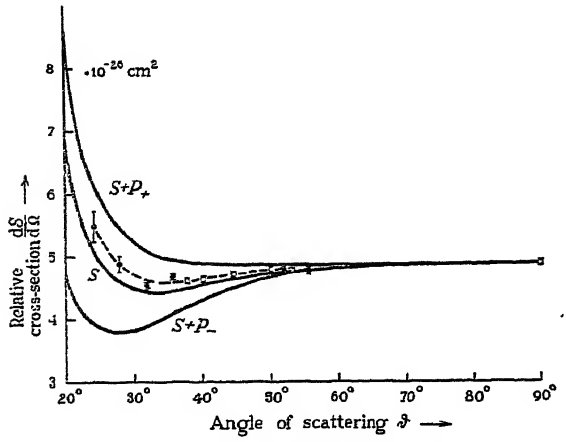

Fig. 7.131. Scattering of 10 MeV protons by protons (WILSON [47*b*]. Comparison of measured relative cross-section $\sigma(\Theta)$ with theory. (The data are adjusted to the theoretical value of the absolute cross-section at $\Theta = 45°$ *i.e.* $\vartheta = 90°$.)
Curve *S*. theoretical *S*-wave scattering.
Curve $S + P_+$: theoretical $S + P$ scattering assuming repulsive P-potential.
Curve $S + P_-$: same, assuming attractive P-potential.
The depth assumed for the P-potential is not clearly specified.

7.14. *Range of potential from disintegration "stars".* An interesting check on the value derived for the range of proton-proton force from the analysis of the scattering experiments has come from quite a different quarter. The disintegration "stars" produced in a photographic emulsion by the impact of cosmic radiation * afford indirect evidence of nucleon scattering for energies of a totally different order of magnitude from those that can be produced in the laboratory. In fact, the tracks forming such stars are interpreted as due mostly to protons knocked out of a heavy nucleus (Ag or Br) of the emulsion by the impact of a nucleon of very high energy ($\approx$ 200 MeV and more). The energy distribution of these protons will primarily be determined by the cross-section for collisions between the impinging fast nucleon and the constituent particles of the nucleus, and this cross-section will in its turn depend sensitively on the range of the interaction between the colliding nucleons: we thus have here, as pointed out by HEISENBERG [37], the possibility of a rather precise determination of the range.

A statistical analysis, performed by ORTNER [40], of empirical material obtained under homogeneous conditions **, leads to a very simple exponential law for the average number $\bar{n}_0(E_0)$ of star particles of energy $\geq E_0$:

$$\log_{10} \bar{n}_0 (E_0) \approx \text{const} - (0{,}0252 \pm 0{,}0045)\, E_0 \qquad (21)$$

(E_0 expressed in MeV)

We shall carry out the comparison with theory for a law of force of the Gauss type ***. It may be shown (*12.35*) that $\bar{n}_0(E_0)$ is proportional to a certain function f_G of the range of the force, which in the case considered

* The subject of disintegration stars is reviewed by BAGGE [43].

** New material, published by PERKINS [47], fits Ortner's formula fairly well for $E_0 \approx 20 \ldots 80$ MeV.

*** As observed in *12.33*, Heisenberg's theory is unfortunately not applicable in its present form to the case of the meson potential.

may be approximated by just an exponential function of the same form as (21), viz. (*12.36–43*)

$$\log_{10} f_G \approx \text{const} - \left[0{,}0288\left(\frac{1}{\varkappa d}\right)^2 + 0{,}0096\left(\frac{2r_0}{d}\right)^2\right] E_0 \qquad (22)$$

(E_0 expressed in MeV).

Adopting as before (*2.22–7*) $r_0 = \frac{1}{2}d$, we immediately get from (21) and (22)

$$\frac{1}{\varkappa d} = 0{,}72 \pm 0{,}1,$$

i.e.

$$\varkappa^{-1} = (2{,}0 \pm 0{,}3) \cdot 10^{-13}\ \text{cm},$$

in excellent agreement with the value resulting from the proton-proton scattering experiments *. Moreover, it will be seen that a change by 10 % of the value chosen for r_0 would only imply a change of about $\pm$ 0,04 in $(\varkappa d)^{-1}$, which is negligible in view of the experimental uncertainty.

7.2. Deuteron formation by proton collisions

Although it has no immediate bearing on our subject, we wish to call attention to the reaction

$$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{D} + \beta^+, \qquad (1)$$

in which two protons combine to form a deuteron, the extra charge being emitted as a lepton field (positon and neutrino). This process, which is of considerable astrophysical importance, has been treated by BETHE and CRITCHFIELD [38]. We may restrict ourselves to the initial state in which the two-proton system has no orbital momentum, for this will be the most favorable for the envisaged proton combination; on account of the exclusion principle, the spins of the protons must then be antiparallel, so that we have to do with a β^+ transition from a ^1S state of the two-proton system to the ^3S ground-state of the deuteron. Assuming the interaction between nucleons and lepton field to be of the Gamow-Teller type (*A 1.11*), such a transition, which belongs to class (*b*) of *A 1.12*, is an allowed one. (On Fermi interaction, it would be forbidden; cf. *A 1.132*.) The cross-section $\mathcal{S}_d$ for the formation of a deuteron by two protons of relative velocity v may then be written in the form

$$\mathcal{S}_d = \frac{1}{4} \cdot \frac{1}{v} \cdot \frac{1}{\tau_0} I(W) |G|^2, \qquad (2)$$

the factor $\frac{1}{4}$ representing the *a priori* probability of the protons having antiparallel spins, while the expression for the probability per unit time of the β^+ transition is taken over from *A 1.11* with the notations explained there. Here one has

$$W = \frac{2M_p - M_d}{m} = 1{,}8 \text{ whence } I(W) = 0{,}132, \qquad (3)$$

by (*A 1.11–3,4*).

* BAGGE's [43] contention that the present method would yield the most precise range determination can obviously not be substantiated.

The matrix-element $|G|^2$ can be decomposed into a factor arising from the summation over spin and isotopic variables, the value of which, according to table *A 1.132*, is 2, and another factor containing the integration over the relative space coordinates; in the latter, angular integration simply yields 4π, so that

$$|G|^2 = 2 \cdot (4\pi)^2 \left| \int_0^\infty R_0(r) \cdot {}^1L_\varepsilon^{(0)}(r)\, dr \right|^2. \tag{4}$$

In this formula, R_0 is the radial part of the deuteron eigenfunction, ${}^1L_\varepsilon^{(0)}$ that of the two-proton system of kinetic energy ε. The latter has the asymptotic form

$${}^1L_\varepsilon^{(0)}(r) \simeq \frac{\sqrt{2}}{k} e^{i\delta} \cos\delta \, [L_0^{(\mathrm{reg})}(r) + L_0^{(\mathrm{irr})}(r) \operatorname{tg}\delta], \tag{5}$$

the L's on the right being defined by (*7.11*–3). In fact, (5) is just the asymptotic form of the $l = 0$ component of the normalized solution of the two-proton problem set up in *7.11*, multiplied by $e^{-i\gamma^{(0)}}/\sqrt{2}$; the phase factor $e^{-i\gamma^{(0)}}$ is without significance, but the other factor $1/\sqrt{2}$ is necessary because the solution used in *7.11* was normalized to 2, while we here want a wave-function normalized to unity (for a proton pair).

Let us assume, as effective potentials, wells of the same width $D = d$ and of depths ${}^3J_0^{(0)}$, ${}^1J_{-1}^{(0)}$ given in tables *6.432* and *7.13*, respectively. The functions R_0, ${}^1L_\varepsilon^{(0)}$ can then be written down explicitly and the integration in (4) performed. The essential velocity dependence of ${}^1L_\varepsilon^{(0)}$, for small values of v, is given by a factor

$$\sqrt{2\pi\alpha e^{-2\pi\alpha}} \quad , \quad \alpha \equiv e^2 c/\hbar v,$$

contained in $L_0^{(\mathrm{reg})} + L_0^{(\mathrm{irr})} \operatorname{tg}\delta$, the square of which expresses the penetrability of the Coulomb potential barrier. The final result can be written

$$\left| \int_0^\infty R_0(r) \cdot {}^1L_\varepsilon^{(0)}(r)\, dr \right|^2 = 2\alpha e^{-2\pi\alpha} ({}^3a_0)^3 \, A^2, \tag{6}$$

where ${}^3a_0$ is given by (*6.21*–14) and A^2 is a complicated function of D, ${}^3J_0^{(0)}$, ${}^1J_{-1}^{(0)}$ and ε_0, representing the sum of the probabilities for the reaction to take place when the two protons are at different mutual distances. The main contribution to A^2 is found to come from the distances $\approx {}^3a_0$ and yields, with the adopted numerical values of the parameters, $A^2 \approx 8{,}1$.

The cross-section (2) finally takes the form

$$S_d = 2\,(2\pi\alpha)^2 e^{-2\pi\alpha} \cdot A^2 \cdot \frac{2}{\tau_0} I(W) \frac{\hbar}{e^2 c} {}^3a_0 \cdot ({}^3a_0)^2 : \tag{7}$$

the meaning of the two first factors has already been explained; the next one, $\frac{2}{\tau_0} I(W) \frac{\hbar}{e^2 c} {}^3a_0$, is just the chance of the positon emission taking place during the time taken by a proton of velocity $e^2c/\hbar$ to go through the distance ${}^3a_0$. Inserting the above numerical values, and taking (*A 1.21*–4) $\tau_0 \approx 3 \cdot 10^3$ sec, formula (7) yields

$$S_d = \alpha^2 e^{-2\pi(\alpha-1)} \cdot 4 \cdot 10^{-50}\ \mathrm{cm}^2. \tag{8}$$

This result, based on the assumption of free proton collisions, is valid under the conditions prevailing in the interior of the stars of the main sequence. For white dwarfs, however, the effective electrostatic interaction between the closely packed protons becomes very different from the Coulomb potential and ${}^1L_\varepsilon^{(0)}$ takes a quite different form; an estimate of S_d in this case has been obtained by WILDHACK [40]. The astrophysical implications are discussed by WILDHACK [40], CHERTOCK [44] and SCHATZMAN [46] *.

* See, in this connexion, *A 1.22.*

insert: it may be applied to the boundary layer; in the degenerate core

CHAPTER VIII

THE CHARGE INDEPENDENCE OF NUCLEAR INTERACTION

8.1. Comparison of proton-neutron and proton-proton potential

We are now in a position to make a quantitative comparison between the nuclear proton-proton potential, as determined in the preceding Chapter, and the potential between proton and neutron in the same (viz. the ^1S) configuration of the two-nucleon system. We begin by assuming that both effective potentials have the same range. This assumption is admittedly not free from uncertainty; in order to test it, the only available calculations are those concerning the well and meson potentials, of which the latter (*7.13, 8.33*) are quite inconclusive. In the case of the well potential (*6.431*), if we took the proton-neutron scattering results at their face value, we should have to adopt a range of proton-neutron potential essentially smaller than that of the proton-proton nuclear force. But, as already observed, the evidence is not so precise as to exclude the possibility of both ranges being equal. Once we accept this equality, however, we meet (as a comparison of the tables *6.432* and *7.13* will show) with the striking feature that in all cases investigated, the strengths of both potentials are also about the same. More precisely, as shown by table *8.1*, the proton-

8.1. *Comparison of proton-proton and proton-neutron potential*

Type of potential	$\frac{{}^1J_0^{(0)}}{{}^1J_{-1}^{(0)}} - 1$
Well	+ 0.039
Morse	+ 0,033
Gauss	+ 0,034
Meson $\frac{M_m}{m} = 200$	(− 0,043)
Meson $\frac{M_m}{m} = 248$	+ 0,023
Meson $\frac{M_m}{m} = 278$	+ 0,012
Meson $\frac{M_m}{m} = 326$	+ 0,017

neutron potential seems to be about 1 ... 4 % larger than the other. The approximate constancy of this ratio for widely different shapes of the potential would perhaps testify to the reality of the difference; but a definite opinion on this point would certainly be premature *. At any rate,

* The conclusion reached by BREIT *et al.* [39*b*] and by THAXTON and MONROE [40*a*] as to the *exact* equality of both potentials was based on Simons' inadequate value of the proton scattering cross-section for zero energy neutrons (*6.22*).

such a difference would be a secondary effect, which we may neglect to a first approximation.

Assuming further (*3.3*) the neutron-neutron potential to be the same as the proton-proton one, we are thus led to the conclusion that *in even singlet configurations (type* ^1S*), the nuclear interaction between two nucleons is independent of the charges of these nucleons* (at any rate to a first approximation). On the other (^3P i.e. odd triplet) type of configurations compatible with the exclusion principle for nucleons of like charge (*4.331*), we have no clear-cut empirical information (*7.13, 7.131*). But it can easily be proved that *any operator of nuclear potential, non-vanishing and charge independent in even singlet configurations, has the same property in all configurations that are at all possible for nucleons of like charge*, i.e. all "charge triplet" configurations. In fact, the matrix-elements of the nuclear interaction operator (*4.331–26*) in such a configuration have, with respect to the isotopic quantum numbers, the diagonal form $(m_t;\ \tau = 3 \mid \ldots \mid m_t;\ \tau = 3)$; and the charge independence property is simply expressed by saying that these matrix-elements are independent of m_t: in other words, the interaction operator itself (with exclusion of the Coulomb repulsion term) is invariant for *all* rotations in isotopic space. This, again, means that all its terms either are independent of the isotopic variables $\tau^{(1)}$, $\tau^{(2)}$, or involve these variables through the only rotation invariant combination $\tau^{(1)}\tau^{(2)}$. Clearly such an operator,

$$V_{\text{nucl}} = \mathcal{A} + \mathcal{B}\,\tau^{(1)}\,\tau^{(2)} \tag{1}$$

($\mathcal{A}$ and $\mathcal{B}$ being functions of position and spin coordinates), is charge independent in any configuration.

It must be observed, however, that this does not at all mean that the general potential operator is necessarily of the form (1), i.e. charge independent in any possible configuration. In the first place, the charge independence in ^1S configurations does not tell us anything about any possible term of non-central interaction: for the expectation value of such a term would vanish in a ^1S state, irrespective of its dependence on the isotopic variables; and on the other hand, the derivation of the numerical value of the potential strength $^1\mathcal{J}^{(0)}$ from the experimental data on slow neutron scattering, and, therefore, the establishment of the charge independence property of even singlet states, is not appreciably affected by the very small modification of the triplet scattering cross-section brought about by non-central forces (*16.13*). But even if we keep to central interactions, the addition to the interaction potential of any term involving the factor $3 + \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$, the expectation value of which vanishes in singlet states, does not impair the charge independence property of such states. Through the occurrence of additional terms containing factors of these types, and depending in an arbitrary way on the isotopic variables, the nuclear potential might be charge dependent in ^3P states. This logical

possibility cannot be dismissed by any physical argument; but it does not seem to correspond to any simple feature of nuclear field theory (an example * will be treated in *16.311*). We shall therefore, in the following, adopt as a general form of dependence of the nuclear potential on the isotopic variables the expression (1) above, which expresses the *charge independence property* of the potential in all possible configurations. One should, however, not lose sight of the fact that this form is only a plausible extrapolation, and that even in the case of even singlet states its empirical foundation leaves room for small deviations.

8.2. General form of central nuclear potential

Exactly in the same way as the charge independence of the nuclear potential restricts its dependence on the isotopic variables to the form (*8.1*–1), the assumption of central interactions further limits each one of the functions $\mathcal{A}$ and $\mathcal{B}$ to the type $a + b\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$, a and b being functions of the distance only: indeed, any non-central coupling being excluded, there remain just the rotation invariant spin functions 1 and $\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$. The most general expression of the central potential would thus involve four independent functions for the distance; but a further assumption will from now on be introduced, viz. that the law of distance dependence is the same for the four possible types of dependence on spin and isotopic variables. This assumption is, of course, made primarily for the sake of simplicity. But the consistency of the results obtained on this basis in the discussion of even (^3S and ^1S) configurations (*6.431, 6.432, 8.1*) may be considered to afford some partial justification for it. The nuclear potential may finally be written in the form

$$\begin{gathered} V_{\text{nucl}} = J(r)\,\mathcal{O} \\ \mathcal{O} = a_0 + a_\sigma\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)} + a_\tau\,\tau^{(1)}\tau^{(2)} + a_{\sigma\tau}\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\tau^{(1)}\tau^{(2)}, \end{gathered} \tag{1}$$

the a's being *constant* coefficients.

We may also decompose the potential (1) into a combination of the four types of exchange interactions (*4.341*–34). Remembering that (*4.13*–26)

$$\vec{\sigma}^{(1)}\vec{\sigma}^{(2)} = 2P_\sigma - 1 \quad , \quad \tau^{(1)}\tau^{(2)} = 2P_\tau - 1 \tag{2}$$

and consequently

$$\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\tau^{(1)}\tau^{(2)} = 4P_\sigma P_\tau - 2P_\sigma - 2P_\tau + 1, \tag{3}$$

we may in fact write

$$\mathcal{O} = W + BP_\sigma - HP_\tau - MP_\sigma P_\tau, \tag{4}$$

* See also HULTHÉN [43*b*], § 4.

with

$$\begin{aligned} W &= a_0 - a_\sigma - a_\tau + a_{\sigma\tau} \\ B &= 2(a_\sigma - a_{\sigma\tau}) \\ H &= 2(a_{\sigma\tau} - a_\tau) \\ M &= -4a_{\sigma\tau}. \end{aligned} \tag{5}$$

From these formulae, it appears that the absence of any dependence on spin and isotopic variables is characterized by

$$a_\sigma = a_\tau = a_{\sigma\tau} \tag{6}$$

(there is then only an ordinary and a Majorana interaction), so that the occurrence of such a dependence is revealed by unequal values of these parameters.

Another mode of decomposition consists in expressing $\mathcal{O}$ as a combination of the effective potentials for the four different types of configurations ^3S, ^1S, ^3P, ^1P, which, on account of (*4.331–23*), may be characterized by the eigenvalues ± 1 of P_σ and P_τ; thus the effective potential for the ^3S configurations will depend on spin and isotopic variables through the combination $\frac{1}{2}(1+P_\sigma)\cdot\frac{1}{2}(1-P_\tau)$. The operator $\mathcal{O}$ accordingly takes the form

$$\begin{aligned} \mathcal{O} = -\tfrac{1}{4}[&(1+P_\sigma)(1-P_\tau)\,{}^3s + (1-P_\sigma)(1+P_\tau)\,{}^1s \\ &+ (1+P_\sigma)(1+P_\tau)\,{}^3p + (1-P_\sigma)(1-P_\tau)\,{}^1p], \end{aligned} \tag{7}$$

with

$$\begin{aligned} -{}^3s &= a_0 + a_\sigma - 3(a_\tau + a_{\sigma\tau}) \\ -{}^1s &= a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau}, \end{aligned} \tag{8}$$

$$\begin{aligned} -{}^3p &= a_0 + a_\sigma + a_\tau + a_{\sigma\tau} \\ -{}^1p &= a_0 - 3a_\sigma - 3(a_\tau - 3a_{\sigma\tau}). \end{aligned} \tag{9}$$

It will be convenient to normalize the constants by taking for $J(r)$ the *absolute value* of the effective ^3S potential. This means ${}^3s = 1$; the constant ${}^1s$ will further be equal to the ratio

$$q = {}^1J_0^{(0)}/{}^3J_0^{(0)}, \tag{10}$$

which, according to table *6.432*, is remarkably insensitive to the law of force adopted for $J(r)$, and roughly equal to 0,6. Formula (7) then becomes

$$\begin{aligned} \mathcal{O} = -\tfrac{1}{4}[&(1+P_\sigma)(1-P_\tau) + (1-P_\sigma)(1+P_\tau)\,q \\ &+ (1+P_\sigma)(1+P_\tau)\,{}^3p + (1-P_\sigma)(1-P_\tau)\,{}^1p]; \end{aligned} \tag{11}$$

we also re-write the formulae (8):

$$\begin{aligned} -1 &= a_0 + a_\sigma - 3(a_\tau + a_{\sigma\tau}) \\ -q &= a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau}. \end{aligned} \tag{12}$$

The parameters ${}^3p$, ${}^1p$ pertaining to the odd configurations cannot be directly given numerical values. We shall presently show how experiments on scattering of fast neutrons by protons, when extended to sufficiently high energies, will eventually provide the information needed to fix the odd-state potentials (*8.33*). It will appear later (*14.12*) that the analysis of the scattering of fast neutrons or protons by deuterons may serve the same purpose. But for the moment, we must make use of less direct arguments: the general saturation requirements for heavy nuclei (*2.22*) impose on the interaction parameters further limitations, which practically amount to their complete determination (*11.33*).

8.21. *Traces of operators related to the nuclear potential.* In a later section, we shall need the expressions for the traces of the operators $\mathcal{O}$, $\mathcal{O}P_\sigma P_\tau$, $\mathcal{O}^2$ and $\mathcal{O}^2 P_\sigma P_\tau$. They are easily found with the help of elementary properties of dichotomic variables (*4.13*). From (1) one immediately gets

$$\begin{aligned} &\operatorname{tr}\mathcal{O} = 16\,a_0, \quad \operatorname{tr}\mathcal{O}P_\sigma P_\tau = 4\,[a_0 + 3a_\sigma + 3\,(a_\tau + 3a_{\sigma\tau})], \\ &\operatorname{tr}\mathcal{O}^2 = 16\,[a_0^2 + 3a_\sigma^2 + 3\,(a_\tau^2 + 3a_{\sigma\tau}^2)]. \end{aligned} \tag{13}$$

To calculate $\operatorname{tr}\mathcal{O}^2 P_\sigma P_\tau$, it is convenient to start from (11); since

$$(1+P)(1-P) = 0, \quad (1 \pm P)^2 = 2\,(1 \pm P), \tag{14}$$

one has

$$\mathcal{O}^2 = \tfrac{1}{4}\,[(1+P_\sigma)(1-P_\tau) + q^2\,(1-P_\sigma)(1+P_\tau) + {}^3p^2\,(1+P_\sigma)(1+P_\tau) + {}^1p^2\,(1-P_\sigma)(1-P_\tau)]. \tag{15}$$

The expressions for $\mathcal{O}\,P_\sigma P_\tau$, $\mathcal{O}^2\,P_\sigma P_\tau$ differ from those for $\mathcal{O}$ and $\mathcal{O}^2$, respectively, only by the sign of the first two terms. We thus obtain the desired traces in a very symmetrical form:

$$\begin{aligned} \operatorname{tr}\mathcal{O} &= -3\,(1+q) - (9\,{}^3p + {}^1p) \\ \operatorname{tr}\mathcal{O}P_\sigma P_\tau &= \;\;3\,(1+q) - (9\,{}^3p + {}^1p) \\ \operatorname{tr}\mathcal{O}^2 &= \;\;3\,(1+q^2) + 9\,{}^3p^2 + {}^1p^2 \\ \operatorname{tr}\mathcal{O}^2 P_\sigma P_\tau &= -3\,(1+q^2) + 9\,{}^3p^2 + {}^1p^2. \end{aligned} \tag{16}$$

8.3. Charge independent interaction and meson field

8.31. *Neutral and symmetrical meson theories.* The charge independence property of nuclear interaction has an important consequence for the meson field theory of this interaction: it implies that — besides the charged meson fields, the production of which by nuclear systems is testified by the phenomena of cosmic radiation — there must also occur (at least virtually, in the sense of *1.321*) neutral meson fields *. Indeed, to a first approximation, a charged meson field will only give rise to an interaction between a proton and a neutron; this is a simple consequence of the

* This has first been pointed out by H. FRÖHLICH, W. HEITLER and N. KEMMER, *Proc. R. S.* **A 166**, 154. 1938.

principle of conservation of charge, as a (virtual) emission or absorption of a charged meson is necessarily accompanied by a transition from proton to neutron state or vice-versa. A force between two nucleons of like charge will first appear in the next approximation, as the result of a double virtual process of meson emission and absorption, and will consequently be of a smaller order of magnitude *.

It will be necessary to consider somewhat more closely the formal aspect of this situation. As is well-known **, the description of a charged field involves two sets of Hermitian components, the relation of which to the charge and current distribution is completely fixed by their properties with respect to gauge transformations. Such a transformation is defined by the addition to the electromagnetic potential of the gradient of an arbitrary function:

$$\frac{e}{\hbar} A_i \to \frac{e}{\hbar} A_i + \frac{\partial \alpha}{\partial x^i}. \tag{1}$$

For our present purpose, it will suffice to consider the simplest case of a meson field of spin 0, represented by scalar or pseudoscalar components (*1.32*); insofar as the charge properties of the field are concerned, the following considerations apply just as well to spin 1 mesons, which require for their representation sets of components possessing vector or pseudovector properties. Let, therefore, ψ_1, ψ_2 be the Hermitian components of a charged meson field of spin 0. It will be convenient to regard them as the orthogonal projections of a symbolic vector in a fixed plane; when the transformation (1) is performed, these components undergo a linear transformation, which may be described as a rotation of the symbolic vector in the fixed plane through the angle α. This property may be concisely expressed by stating that the non-Hermitian quantity $\psi = \frac{1}{\sqrt{2}} (\psi_1 + i \psi_2)$ is multiplied by the phase factor $e^{i\alpha}$:

$$\psi \to \psi e^{i\alpha}. \tag{1a}$$

It will not be necessary for our argument to explain further how the requirement of invariance of the Lagrangian for the transformation (1), (1a) yields the expression for the charge and current density in terms of ψ and its adjoint $\psi^\dagger$ (or, what amounts to the same, in terms of ψ_1 and ψ_2). The only property of the field quantity ψ which we need is the following: if ψ is expanded in a Fourier series, i.e. in a series of eigenfunctions of the meson states of definite momentum,

$$\psi = \sum_{\vec{p}} \left[b_+ (\vec{p})\, e^{\frac{i}{\hbar} \vec{p}\,\vec{x}} + b_-^\dagger (\vec{p})\, e^{-\frac{i}{\hbar} \vec{p}\,\vec{x}} \right], \tag{2}$$

* One might think that this argument would be restricted to the case of weak coupling (*1.35*), but it has been verified (*A 3.24*) that even a strong coupling theory involving only charged mesons is unable to account for the charge independence of the nuclear interaction.

** See, e.g., MØLLER and ROSENFELD [43].

the Fourier amplitudes $b_+(\vec{p})$, $b_-(\vec{p})$ are such that when operating on the wave-function of any state of the field characterized by definite numbers of mesons of given momenta, they decrease by one the respective numbers of positive and negative mesons of momentum $\vec{p}$; the adjoint operators $b_+^\dagger$, $b_-^\dagger$ accordingly increase these numbers by one. Every term of the expansion (2) of ψ is thus connected with a decrease of the charge of the meson field by one (positive) unit, either by absorption of a positive or emission of a negative meson.

The interaction energy between the meson field and a system of nucleons may be written

$$\mathcal{V}_\psi = -\int (\psi^\dagger s + s^\dagger \psi)\, dv, \tag{3}$$

the functions s, $s^\dagger$ of the nucleon variables being just the source densities which enter into the fundamental field equations of the form (*1.32*–10) for ψ and $\psi^\dagger$. Now, it is clear that in order to satisfy the requirement of charge conservation of the total system of nucleons and mesons for every emission or absorption process, the coefficients of $\psi^\dagger$ and ψ in (3) must be linear combinations of the operators $\Pi^{(i)\dagger}$ and $\Pi^{(i)}$, respectively, expressing that the change of charge of the meson field is compensated by an accompanying transition of the i-th nucleon from the proton to the neutron state, or vice-versa (*4.12*–24). In other words, we may write the density s in the form

$$s = \sqrt{2} \sum_i \Pi^{(i)\dagger} s^{(i)}, \tag{4}$$

the symbol $s^{(i)}$ denoting a certain function of the space and spin variables of the i-th nucleon, and also of the point of space-time at which the density s is taken, while (*4.12*–24) $\Pi^{(i)\dagger} = \frac{1}{2}(\tau_1^{(i)} + i\tau_2^{(i)})$. In all cases, the $s^{(i)}$ are Hermitian, so that the source densities of the Hermitian components ψ_1, ψ_2 are

$$s_1 = \sum_i \tau_1^{(i)} s^{(i)}, \quad s_2 = \sum_i \tau_2^{(i)} s^{(i)}, \tag{5}$$

respectively; the interaction energy (3) takes the form

$$\mathcal{V}_\psi = -\int (\psi_1 s_1 + \psi_2 s_2)\, dv. \tag{6}$$

The formulae (5) exhibit an extremely simple and direct connexion between the symbolic isotopic space (*4.22*) and that which has just been introduced to express the gauge transformation properties of the charged field. In particular, the gauge invariance of the expression (6) is secured by the fact that the dichotomic variable $\tau_\perp$, whose components are τ_1, τ_2, is free to rotate in its plane (*4.12*).

When we consider the interactions between nucleons under circumstances prevailing in atomic nuclei or in scattering experiments involving

energies not much higher than those hitherto used, we may to a first approximation disregard all dependence of the source density on the time variations of the various coordinates of the nucleons and accordingly neglect the retardation of the force transmission through the meson field; this approximation will yield the *static* interaction. Higher terms form the *non-static* part of the interaction, which includes not only the effect of the *translation* motions of the nucleons, but also those of the *precession of their spins* and of the *exchanges of electric charge* between them (transitions between proton and neutron states, mathematically described by precessions of the isotopic vectors $\tau^{(i)}$ in isotopic space). From the expression (3) or (6) one obtains by the usual methods of field theory (*15.31*), with the help of the Green's function (*1.31, 1.32–9*) $\varphi = \frac{1}{r} e^{-\varkappa r}$, the total static interaction energy of a system of nucleons in the form of a sum of interactions between pairs

$$\overset{\circ}{V}{}_{\psi}^{(12)} = 2\, \mathsf{T}^{(12)} \int \overset{\circ}{s}{}^{(1)}(P')\, \varphi(P'; P)\, \overset{\circ}{s}{}^{(2)}(P)\, dv_P\, dv_{P'}; \tag{7}$$

in this expression, $\overset{\circ}{s}{}^{(i)}(P)$ denotes the static approximation of the density function $s^{(i)}$ occurring in (4), (5), taken at point P; the *isotopic factor* $\mathsf{T}^{(12)}$, defined by (*4.41–1*), has a non-vanishing expectation value only for a proton-neutron pair. Clearly, then, if static forces are predominant, charged meson fields alone will be unable to account for the charge independence property.

An obvious way of arriving at a law of interaction independent of the charges would be to introduce a purely neutral meson field, the source density of which would simply be independent of the isotopic variables of the nucleons. Since a neutral field is characterized by its invariance with respect to gauge transformations, i.e. rotations in the plane (**1**, **2**) of isotopic space, it can be represented by a Hermitian component ψ_3 directed along the 3-axis of this space. If s_3 is the corresponding source density, the interaction energy of the neutral field with the nucleons will be of the type

$$V_{\psi_3} = -\int \psi_3\, s_3\, dv. \tag{8}$$

The hypothesis just formulated consists in putting, similarly to (5),

$$s_3 \sim \sum_i s^{(i)}, \tag{9}$$

which leads to an interaction between nucleons completely independent of their charges in any approximation. However, it must be stressed that the simplicity of this picture is only apparent: for there are, anyhow, charged meson fields, produced by the nucleons, which will inevitably give rise to a certain additional amount of proton-neutron interaction. One should therefore assume that the source densities of the charged fields are

sufficiently small compared with those of the neutral field to account for the almost complete charge independence of the total interaction deduced from the experiments; the small and uncertain deviation from exact charge independence, exhibited by these experiments, should from this point of view be regarded as real and would indicate an upper limit to the ratio of the magnitudes of the source intensities of charged and neutral fields. It is true that the smallness of the charged meson fields would not come into conflict with any known fact concerning cosmic radiation. Nevertheless, such a predominantly neutral meson theory would be very unsatisfactory indeed, inasmuch as in it the connexion between nuclear forces and cosmic mesons, the most beautiful and significant feature of Yukawa's idea, would be completely lost. In fact, the assumptions made about the mass and other properties of the neutral mesons would be entirely arbitrary, and the whole scheme would be little more than a rather clumsy and roundabout way of putting forward a special form of interaction potential, distinguished from any other only by its unwieldiness. In spite of this objection, we shall in all following discussions retain the *neutral* type of nuclear field theory as a logical possibility to be tested by experiment.

There is, however, an alternative possibility of securing the charge independence of the nuclear potential. It is the so-called *symmetrical* type of meson theory, pointed out by KEMMER [38a]. As we have seen above (*8.1*), the total static interaction will be charge independent if the contribution from the neutral meson field is such as to add to the isotopic factor $2\,\mathsf{T}^{(12)} = \tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)}$ of the charged field interaction (7) the term $\tau_3^{(1)}\tau_3^{(2)}$, completing the scalar product $\tau^{(1)}\tau^{(2)}$. Comparing (5), (6) with (8), we see that this result is achieved by taking

$$s_3 = \sum_i \tau_3^{(i)}\, s^{(i)}. \tag{10}$$

In this combination of charged and neutral fields, the three Hermitian components ψ_1, ψ_2, ψ_3 enter in an entirely symmetrical way; we may regard them as the three projections of a vector ψ in isotopic space, and the source densities are likewise components of a symbolic vector

$$s = \sum_i \tau^{(i)}\, s^{(i)}. \tag{11}$$

The neutral and the symmetrical theories correspond so to speak to opposite extreme cases of the general law of interaction (*8.1*–1), viz. $\mathcal{B} = 0$ and $\mathcal{A} = 0$. From the point of view of meson theory, these two extreme cases are the only simple ones, for the general expression for the interaction, with $\mathcal{A}$ as well as $\mathcal{B} \neq 0$, could only be obtained by assuming two independent kinds of neutral mesons, which could only be distinguished from each other by the different ways in which they interact with nucleons, viz. according as their source density is of the form (9) or (10). Indeed, the most general function of the isotopic variables which might occur in the source density of a neutral field would be a linear combination of operators T_+ and T_-, which express that the nucleon does not change its proton or

neutron state when emitting or absorbing a neutral meson. This amounts to a source density of the form

$$s = \sum_i (a + b\,\tau_3^{(i)})\, s^{(i)}, \tag{12}$$

in which the coefficients a and b are not necessarily real. This leads to an isotopic factor in the law of interaction

$$|a|^2 + |b|^2\,\tau_3^{(1)}\tau_3^{(2)} + (ab^* + a^*b)(\tau_3^{(1)} + \tau_3^{(2)}),$$

and the condition that the interaction be symmetrical with respect to the charges implies that the coefficient of $\tau_3^{(1)} + \tau_3^{(2)}$ should vanish. Unless, therefore, either a or b vanishes, the source density (12) would be essentially non-Hermitian, which means that we should have two independent Hermitian field functions, representing two kinds of neutral fields differently coupled to the nucleons. This would, of course, be a very artificial state of affairs.

To sum up, we see that while the introduction of neutral meson fields besides the charged ones provides the possibility of deriving a law of static interaction between nucleons of the general charge-independent form (*8.1*–1), the special cases of *purely neutral* and *symmetrical* theories, characterized by the respective isotopic factors 1 and $\tau^{(1)}\tau^{(2)}$, appear more natural from the point of view of meson theory.

8.311. *Instability of the neutral meson.* If there is any truth in the field conception of nuclear forces and their approximate charge independence, a large proportion of the mesons produced by the primary cosmic radiation in the atmosphere must be expected to be neutral. If the life-time of the neutral mesons were comparable with that of the charged ones, they could be detected thanks to a *charge transfer effect*, by which a neutral meson can emerge as a charged one from a collision with a nucleus, according to the scheme *

$$\mu^0 + p \to \bar{n} + \mu^+ \quad , \quad \mu^0 + n \to p + \mu^- . \tag{13}$$

Such non-ionizing agents, capable of producing ionizing secondaries, can in principle be disclosed by a *coincidence method*, the main features of which go back to Rossi, and which has been applied to the present problem both by ROSSI and his collaborators [40] and by NISHINA and BIRUS [41*b*]. We shall outline the somewhat simpler arrangement used by the latter (fig. *8.311*–1): the rates of coincidences of two counters A and B are compared when a lead absorber (sufficiently thick to rule out effects due to photons) is placed either between the counters (position (1)) or above them (position (2)); a non-ionizing agent giving rise to penetrating charged secondaries in the absorber would indeed manifest itself by additional coincidences when the absorber is in position (2). However, this inter-

* The symbols μ^0, $\mu^\pm$ denote neutral and charged mesons, respectively.

pretation of a possible positive effect cannot be accepted without a control provided by the set of counters C placed above the apparatus and covering the whole area of incidence of the rays capable of passing through both counters A and B: clearly, if the difference in twofold coincidence is due to a neutral agent, the rate of threefold coincidences of the counters A, B

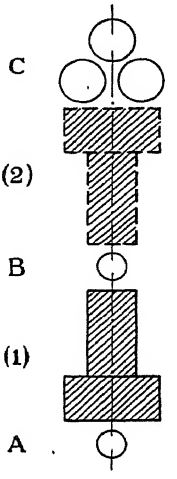

Fig. 8.311–1. Coincidence method of detection of non-ionizing agents producing ionizing secondaries.

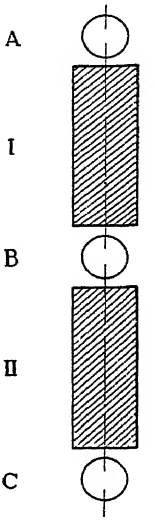

Fig. 8.311–2. Principle of arrangement for detecting charge transfer effect of meson.

and C recorded during the same period as the twofold coincidences, should be the same for both positions of the absorber. In the actual experiments, both the rates of threefold and twofold coincidences were larger with the absorber in position (2), the increase being nearly the same for threefold and twofold coincidences; there was consequently hardly any change in the rate of anticoincidences (i.e. coincidences of A and B not accompanied by a discharge of C) when the absorber was moved from position (1) to position (2). This result strongly points to some other origin of the effect than the production of charged secondaries by non-ionizing rays; and previous experiments, in which the precaution of the control counters C was not taken, lose any significance. The same conclusion is reached by ROSSI *et al.* [40], who further analyze the nature of the spurious effects due to meson scattering and shower production in the neighbourhood of the apparatus, and confirm the predominance of such effects by cloud chamber photographs *.

The charge transfer effect (13), together with its inverse, gives rise to an alternation of the charged and neutral states of a meson during its passage through matter. This phenomenon can be studied by an arrangement of counters and absorbers, very simple in principle (fig. *8.311–2*). A charged meson, detected by counter A, may in absorber I go over to the neutral state and traverse counter B without discharging it; in a second

* See further TRUMPY and BJØRDAL [42], and CLAY and LEVERT [46].

absorber *II*, identical to *I*, it will have equal chance to recover its charge and cause a coincidence of counter *C* with counter *A*. The mean free path of a meson with respect to charge transfer will therefore (on the assumption of sufficient stability of the neutral state) be very simply related to the ratio of the rates of anticoincidences in which *A* and *C* but not *B* are discharged to the twofold coincidences of *A* and *C*. The experiment has been performed by NISHINA *et al.* [41*a*, *b*] with negative result: within the experimental error, no genuine anticoincidences could be discerned.

The conclusion to be drawn from these unsuccessful attempts at detecting neutral mesons * is that the life-time of these particles must be considerably shorter than that of charged mesons. This would seem to imply, from a theoretical point of view, that the interaction between mesons and lepton pairs should be highly unsymmetrical with respect to charged and neutral mesons: the coupling of the neutral meson field to the lepton field should be much stronger than that of the charged meson field, — in a way somewhat analogous to the situation in the neutral theory of nuclear interaction (in which the coupling with the nucleons is assumed to be much stronger for neutral than for charged mesons). There is in fact no objection to such an assumption; in particular, as has been stressed by PAIS [45], we have no reason to expect that the very small interactions between two leptons on account of their meson fields should exhibit any particular symmetry with respect to the charges of the leptons, since the relativistic effects, which will in this case predominate because of the smallness of the lepton masses, will (even if the source densities are symmetrical with regard to charge) be quite different for electrons and neutrinos owing to the large difference between their masses.

According to this view, the short-lived neutral mesons would decay into a pair of neutrinos (and thus be utterly lost to observation) or into a pair of electrons of both signs. The latter would have sufficient energy to initiate showers; an indirect detection of neutral mesons could therefore eventually be obtained if the analysis of the soft component of cosmic radiation, especially at high altitudes, could be sufficiently improved to allow such showers to be recognized. However, even if the decay of neutral mesons into lepton pairs does not have a larger probability than the corresponding decay of charged mesons, there is still another mode of decay, the estimated probability of which is so large that it may possibly be preponderant in bringing about the rapid disappearance of the neutral mesons. It is a process, pointed out by SAKATA and TANIKAWA [40], by which a neutral meson ultimately decays into a certain number of photons. This process may be described as follows: the meson transforms, by a *virtual* transition, into a pair consisting of a proton and an antiproton; these two virtually annihilate each other with photon emission. The number of

* Prof. Heitler has kindly pointed out to me that the negative results of the experiments just described might simply be due to the fact that the cross-section for the charge transfer effect becomes extremely small for mesons of high energy.

photons emitted is at least two, except in the case of a vector meson, when owing to the special symmetry properties of the matrix-element corresponding to the virtual transformation of the meson into a proton pair, the minimum number of emitted photons is three. These photons will again initiate showers, which should form a not inconsiderable part of the soft component of cosmic radiation.

A very rough estimate of the order of magnitude of the life-time t_0' of the neutral meson, according to the last decay process, is given by

$$\frac{1}{t_0'} \approx \frac{g^2}{\hbar}\left(\frac{e^2}{\hbar}\right)^n c\varkappa, \tag{14}$$

where n ($= 2$ or 3) is the least number of photons emitted, while g represents, as in *1.35*, the order of magnitude of the constants analogous to the electric charge e, occurring in the *nucleon* source densities of the meson field. Numerically (assuming the same mass for the neutral as for the charged mesons), formula (14) gives *

$$t_0' \approx 10^{-18} \ldots 10^{-16}\ \text{sec} \tag{15}$$

(according as $n = 2$ or 3). The life-time corresponding to decay into leptons would depend on the choice of the constants $\breve{g}$, analogous to g for *lepton* source densities. If the same values are adopted for neutral and charged mesons, the life-times will also be the same, i.e. (*1.332*) of the order of 10^{-6} sec. The photon decay of the neutral meson will therefore be the predominant one, unless rather extreme assumptions are made about the order of magnitude of the coupling constants $\breve{g}$ for neutral mesons.

8.32. *Central static interactions on meson theory.* We have seen how the charge independence requirement of the static interaction between two nucleons could be fulfilled either by a purely neutral meson field or by a symmetrical combination of charged and neutral fields. The next question is, whether this static interaction can be reduced to a central potential. This point will, however, be more suitably treated in a later Chapter (*16.42*); at this stage, it will suffice to know that a central interaction will result from a suitable mixture of meson fields of spin 0 and 1, both in the case of a purely neutral and in that of a symmetrical theory. In the notation of (*8.2–1*), this central interaction thus takes either of the following forms:

$$\mathcal{V}^{(12)}_{\text{neut}} = (a_0 + a_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)})\, J(r) \tag{16}$$

$$\mathcal{V}^{(12)}_{\text{symm}} = \tau^{(1)} \tau^{(2)} (a_\tau + a_{\sigma\tau} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)})\, J(r)\,; \tag{17}$$

and we shall now be concerned with the task of confronting the consequences of these two laws of nuclear interaction with relevant experiments.

The main point here is that in both (16) and (17) the parameters a are completely fixed by the properties of the S states of the two-nucleon systems, and that the effective potentials for odd states can therefore

* A detailed calculation by FINKELSTEIN [47] leads to $t_0' \approx 10^{-16} \ldots 2 \cdot 10^{-11}$ sec.

uniquely be deduced from them. A comparison will thus be possible between theoretical predictions and experimental results concerning phenomena in which states of higher orbital momenta play a part. Since there is no such state of binding for the two-nucleon system, we are limited to the investigation of the scattering of sufficiently fast nucleons (neutrons or protons) by protons. The experimental study of proton-proton scattering has perhaps the advantage of greater accuracy, but the interpretation of the results is complicated by the fact that we have to do with interference of the proper nuclear scattering with the effect of Coulomb interaction; on the other hand, just this circumstance makes it possible to use *relative* measurements of the differential cross-section for a comparison between nuclear and Coulomb forces. The scattering of fast neutrons by protons, though more difficult of approach, yields very direct evidence on the nuclear potential. Especially the angular distribution of the scattered neutrons (or of the recoil protons) discloses by its deviation from isotropy in the barycentric system of reference the contribution of states of higher orbital momenta to the scattering process. It is this effect which will mainly occupy our attention.

It is clear at the outset that the neutral and symmetrical theories must lead to very different results, for the properties of the odd-state potentials are quite different in the two cases *. In the neutral theory, the odd-state potentials are evidently equal to the corresponding ones for even configurations:

$$ {}^{\sigma}V_{\text{neut}}^{(\text{odd})} = {}^{\sigma}V_{\text{neut}}^{(\text{even})} \qquad (\sigma = 1,3). \tag{18} $$

In the symmetrical theory, on the other hand, the occurrence of the factor $\tau^{(1)}\,\tau^{(2)}$ in (17), in conjunction with the correspondence expressed in (*4.331–23*), immediately leads to the relations

$$ \begin{aligned} {}^{3}V_{\text{symm}}^{(\text{odd})} &= -\tfrac{1}{3}\,{}^{3}V_{\text{symm}}^{(\text{even})} \\ {}^{1}V_{\text{symm}}^{(\text{odd})} &= -3\,{}^{1}V_{\text{symm}}^{(\text{even})}. \end{aligned} \tag{19} $$

Qualitatively, we see that *while in a neutral theory all effective potentials are attractive, in a symmetrical theory the potentials are attractive for even configurations and repulsive for odd configurations.* We shall now proceed to a closer discussion of the implications of these properties for the phenomenon of fast neutron scattering by protons.

8.33. *Scattering of fast neutrons by protons.* The differential scattering cross-section is given by the fundamental formula (*6.21–10*), which corresponds to an angular distribution of the general form

$$ d\mathcal{S} = \frac{1}{16\,k^2}\,d\Omega \sum_n ({}^{1}A_n + 3\cdot{}^{3}A_n)\cos^n\vartheta, \tag{20} $$

* The validity of the relations (18), (19) is not restricted to meson theory, but covers all central interactions of the forms (16) and (17), with any distance dependence $J(r)$.

the ${}^{\sigma}A$'s being certain functions of the phases ${}^{\sigma}\delta^{(l)}$. For each value of n, *all* phases give a contribution to the corresponding ${}^{\sigma}A_n$; we may write

$$\begin{aligned} {}^{\sigma}A_n &= \sum_{l+l' \geq n} a_{ll'}\, {}^{\sigma}a^{(ll')}, \\ {}^{\sigma}a^{(ll')} &\equiv 4 \sin {}^{\sigma}\delta^{(l)} \sin {}^{\sigma}\delta^{(l')} \cos \left({}^{\sigma}\delta^{(l)} - {}^{\sigma}\delta^{(l')}\right), \end{aligned} \tag{21}$$

with certain numerical coefficients * $a_{ll'}$. Explicit formulae, including the phases up to $l = 5$, will be found in a paper by PAIS [46]. The calculation of the phases can be carried out by the methods explained in *5.3*; for each value of the energy of the incident neutrons, the ${}^3S$ and ${}^1S$ phases have to be computed by the variational method (*5.31*), the higher phases by the approximate procedure of *5.331* or simply by Born's approximation. The reduced strength ${}^1b$ of the effective ${}^1S$ meson potential can be taken equal to the critical value (*5.231*) ${}^1b \approx 1{,}68$, independent of the value of the meson mass; this only amounts to neglecting the small energy of the virtual ${}^1S$ state (*5.323, 6.23*). The strength ${}^3b$ of the effective ${}^3S$ potential, on the other hand, must be calculated, for each assumed value of the meson mass, by means of formula (*5.231–25*), which, using the value (*6.11–1*) of the energy of the deuteron ground state, takes the form

$${}^3b \approx 1{,}68 + 201{,}2\, \frac{m}{M_m}, \tag{22}$$

with sufficient accuracy.

The main features of the resulting angular distribution, in the case of the

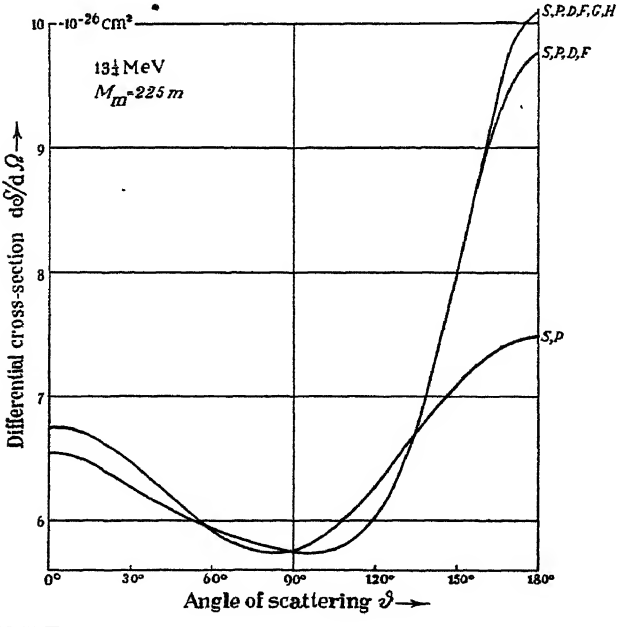

Fig. 8.33–1. Contribution of different orders in the case of a meson mass of 200 m and an energy of 13¼ MeV.

* If n is even, only terms for which l and l' are of the same parity occur in the series (21); if n is odd, there occur only terms with l and l' of different parities.

symmetrical meson theory, are clearly brought out in the figures 8.33–1, 2, 3, due to FRÖHLICH, RAMSEY and SNEDDON [46] *. Fig. 1 shows how the successive inclusion of higher and higher phases in (21) modifies the angular distribution; in particular, it strikingly exhibits the insufficiency of an approximation limited to the P phases. Fig. 2 illustrates the influence

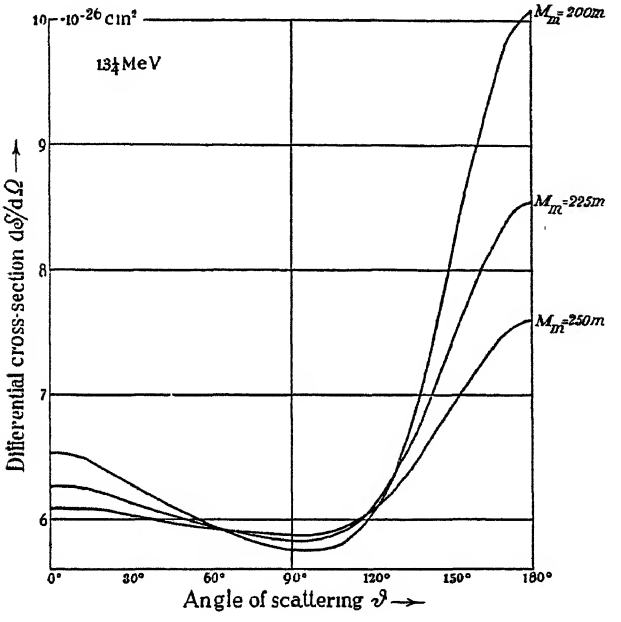

Fig. 8.33–2. Angular distribution of scattering to be expected for 13¼ MeV neutrons.

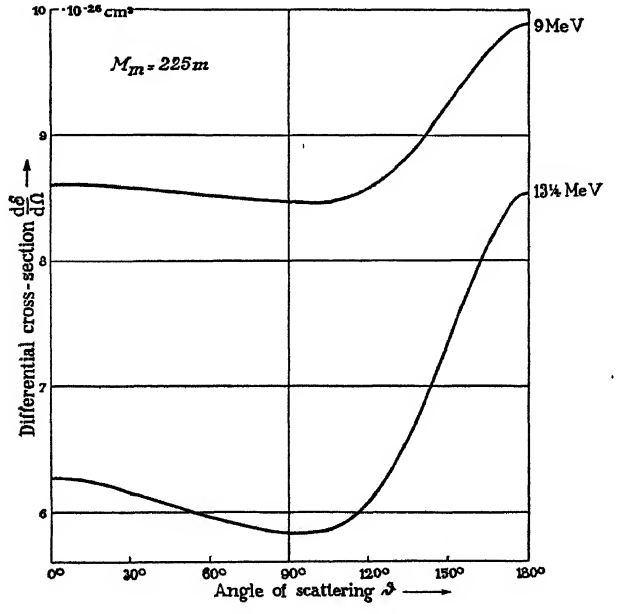

Fig. 8.33–3. Angular distribution of scattering to be expected for 9 MeV and 13¼ MeV neutrons on assumption that meson mass is 225 m.

* See further RAMSEY [47].

of the range of the force (i.e. the value of the meson mass) on the differential cross-section. Finally fig. 3 shows how the deviation from isotropy becomes more and more accentuated with increasing energy of the incident particles: it gives the angular distribution calculated upon the assumption of a meson mass $M_m = 225m$ for the energies 9 and 13,25 MeV. The comparison of these curves with the results of the Bristol experiments (after correction for the "loss" of tracks from the emulsion, as explained in *6.42*) is exhibited by fig. *6.42–3, 4*. It appears from these diagrams that the experiments indeed seem to agree somewhat better with the predictions of symmetrical meson theory than with a purely isotropic distribution; but the deviation from isotropy is still too small at these energies to permit a clear-cut conclusion *.

The theory shows, however, that for only slightly larger energies one can hope that the comparison with observation will enable us (provided we *assume* the nuclear interaction to be mainly central) to decide between a neutral or a symmetrical form of isotopic factor. This results from extensive calculations, carried out by HULTHÉN and PAIS [*46b*], of the *anisotropy ratio*

$$A = S(\pi)/S(\tfrac{1}{2}\pi), \tag{23}$$

which conveniently expresses the main character ("forward" or "backward") of the anisotropy and permits an easier vizualization of its variation with the different parameters of the problem. In terms of the ${}^{\sigma}a^{(ll')}$ introduced in formula (21), the ratio A takes the form (PAIS [46])

$$= \frac{\sum\limits_{m,n}(-1)^{m+n}(2m+1)(2n+1)({}^{1}a^{(mn)}+3\cdot{}^{3}a^{(mn)})}{\sum\limits_{m,n}(-1)^{m+n}(4m+1)(4n+1)\frac{1\cdot3\cdot5\ldots(2m-1)}{2\cdot4\cdot6\ldots2m}\cdot\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}({}^{1}a^{(mn)}+3\cdot{}^{3}a^{(mn)})}.$$

From this formula, a result of extended validity can be obtained upon the assumption that the S phases (and consequently the absolute values of all other phases) are smaller than π, and that all differences $|{}^{\sigma}\delta^{(l)}-{}^{\sigma}\delta^{(l')}|<\pi/2$. In a symmetrical theory, owing to the alternation of attractive and repulsive potentials in even and odd states, one has then (*5.131, 5.132*)

$$(-1)^{m+n}\cdot{}^{\sigma}a^{(m,n)}>0,$$

whence $A>1$: *under the conditions stated, a symmetrical theory leads to a predominantly backward scattering.* A neutral theory, in which all phases are positive, does not imply any such general property of the anisotropy ratio.

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* Similar calculations have been carried out by HULTHÉN [*43a, 44d*] for an energy of 14,5 MeV and a meson mass of 200 m. However, he applied to the b-values for the S states certain corrections which later investigation has invalidated, so that these results cannot be retained.

The calculation of A has been performed for a range of energies from 5 to 25 MeV and two different values of the meson mass, viz. 200 and 300m, in both the symmetrical and the neutral theory. The results are collected, together with some others, concerning the 9 and 13 MeV

8.33. *Scattering of fast neutrons by protons; theoretical results*

Energy MeV	Meson mass M_m/m	Symmetr. theory A	Symmetr. theory S 10^{-24} cm²	Neutral theory A	Neutral theory S 10^{-24} cm²	Reference
5	200	1,07	1,63	1,17	1,65	Hulthén and Pais [46b]
10		1,34	0,97	1,06	1,02	
15		2,01	0,68	0,96	0,74	
20		2,76	0,52	0,80	0,59	
25		3,73	0,43	0,73	0,49	
5	300	1,03	1,60	1,06	1,60	Hulthén and Pais [46b]
10		1,09	0,93	1,05	0,94	
15		1,22	0,65	1,02	0,66	
20		1,42	0,50	0,96	0,51	
25		1,68	0,41	0,90	0,42	
9	200	1,3	1,1	1,09	1,15	Hulthén and Pais [46b]*
	225	1,17	1,03	—	—	Ramsey [47]
	300	1,08	1,06	1,05	1,07	Hulthén and Pais [46b]*
13,5	177	2,58	0,82	0,86	0,89	Pais [46]***
13,25	200	1,70	0,75	1,0**	0,85**	Ramsey [47]
	225	1,45	0,74	—	—	
	250	1,29	0,74	—	—	
	300	1,17	0,76	1,03	0,77	Hulthén and Pais [46b]*
18	225	1,90	0,56	—	—	Ramsey [47]

* Interpolated.
** Interpolated from Hulthén and Pais [46b].
*** The S phases have been computed by Ferretti [43a].

neutrons, in table *8.33*; their characteristic features are clearly exhibited by fig. *8.33–4*. In the table, values of the total cross-section S have also been listed; the comparison with the empirical values collected in table *6.413* is shown by fig. *8.33–5*. While the total cross-section is quite insensitive to the range of the interaction and to its dependence on the isotopic variables, the angular distribution, as expressed by the anisotropy factor, becomes, at higher energies, very markedly different for neutral and symmetrical theories, and in the latter case it also varies appreciably with the meson mass. The study of the angular distribution of very fast neutrons scattered by protons is therefore, as already stated, one of our most powerful means of inquiring into the nature of nuclear forces.

8.34. *Photodisintegration of the deuteron and meson theory.* As explained in *6.51*, the photodisintegration differential cross-section consists

of two terms of different origins, the photoelectric and the photomagnetic effect, which can essentially be distinguished from each other by their different dependence on the angle of ejection of the nucleons with respect to the direction of the impinging photon. In fact, since the photomagnetic

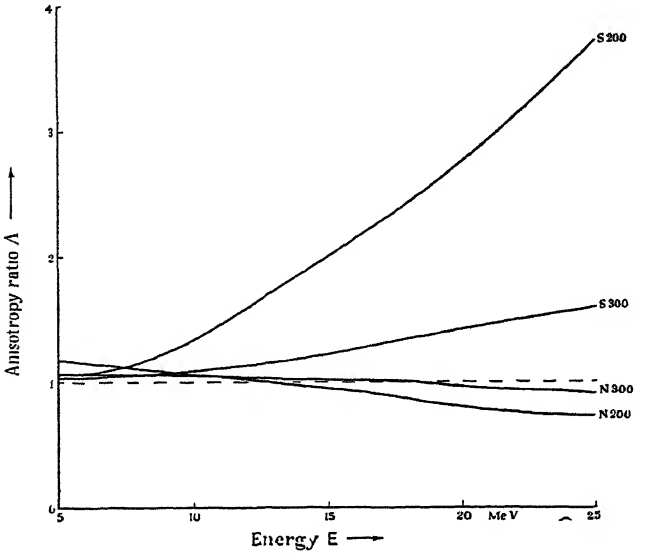

Fig. 8.33–4. Anisotropy ratio, according to different theories, neutral (N) or symmetrical (S), assuming a meson mass $M_m = 200$ or $300\,m$ (as indicated by the number following the symbol N or S).

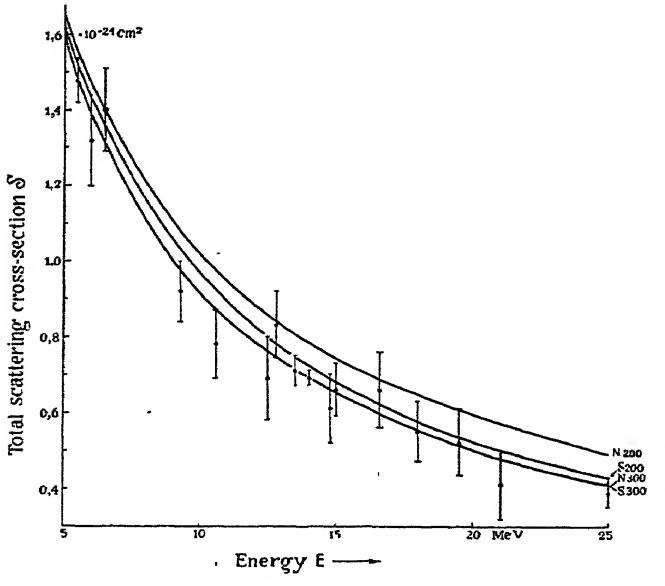

Fig. 8.33–5. Total proton-neutron scattering cross-section according to meson theory. (The curve N_{300} nearly coincides with S_{300}.)

effect is spherically symmetrical (in the barycentric system *), while the photoelectric effect has a $\sin^2\vartheta$-distribution, the ratio of the intensities

* For photons of not too high frequency (and therefore sufficiently small momentum), the barycentric system of reference can be identified with the laboratory system.

of ejected nucleons in directions parallel and perpendicular to that of incidence of the radiation will be given by

$$\mathsf{H} = \frac{d\Phi_{\text{magn}}}{d\Phi_{\text{magn}} + d\Phi_{\text{el}}(\pi/2)} = \frac{\Phi_{\text{magn}}}{\Phi_{\text{magn}} + \frac{3}{2}\Phi_{\text{el}}}. \tag{25}$$

Thus, the measurement of H, together with that of the total cross-section, would give us the contributions Φ_{el} and Φ_{magn} separately. Unfortunately, owing to the smallness of the effects concerned, such measurements are very difficult, and it has not yet been possible to attain the accuracy required for the isolation of Φ_{magn} from the empirical data. All one can do at present is to compare the value of H directly measured, for a given frequency, with that calculated from theory.

This parameter H is essentially suited to give definite indications about the isotopic factor of the interaction potential. In the first place, the eigenfunctions of the ^3P states entering into the expression of the photoelectric cross-section Φ_{el} will be different in a symmetrical and in a neutral theory, because of the different behaviours of the corresponding effective potentials; and this may be expected to affect appreciably the value of Φ_{el}. In the second place the exchange magnetic moment occurring in Φ_{magn}, being obviously determined by the charged meson fields only, will vanish in a neutral theory. In the computations carried out by PAIS [43], the former effect was neglected, and the latter exaggerated owing to a slip in the basic formula *, so that his results are not reliable.

The diagrams published by Pais are here reproduced (fig. *8.34*–1, 2) only to give a general qualitative idea of the relative magnitudes of the two cross-sections Φ_{el} and Φ_{magn} and of their variation with the excitation energy. More accurate calculations have been started by HULTHÉN; for photons of 2,62 MeV energy, he finds, assuming a meson mass $M_m = 200m$:

8.34–1. *Theoretical photodisintegration cross-sections for* $h\nu = 2{,}62$ *MeV*.

	Φ_{el} 10^{-28} cm^2	Φ_{magn} 10^{-28} cm^2	H
Neutral theory	13,76	3,08	0,130
Symmetrical theory	12,07	3,075	0,145

Thus, the exchange effect proves to be quite negligible, and the change in Φ_{el} produces only a relatively small variation of H : a great accuracy would indeed be required to bring it out.

* Dr. Pais has kindly informed me that in the formula on p. 18, l. 10 from bottom, of his paper, the factor $\beta + n$ in the argument of the arctg should be $\beta + n + 1$.

The first experiments on the angular distribution of the ejected particles consisted in investigating cloud chamber tracks of the photo-protons. Neither CHADWICK, FEATHER and BRETSCHER [37] using 2,62 MeV (Th C″) γ-rays, nor RICHARDSON and EMO [38] with 3 MeV γ-rays from the Na (d, p) reaction, could in this way establish with certainty the existence of an isotropic component, but the accuracy was very poor owing

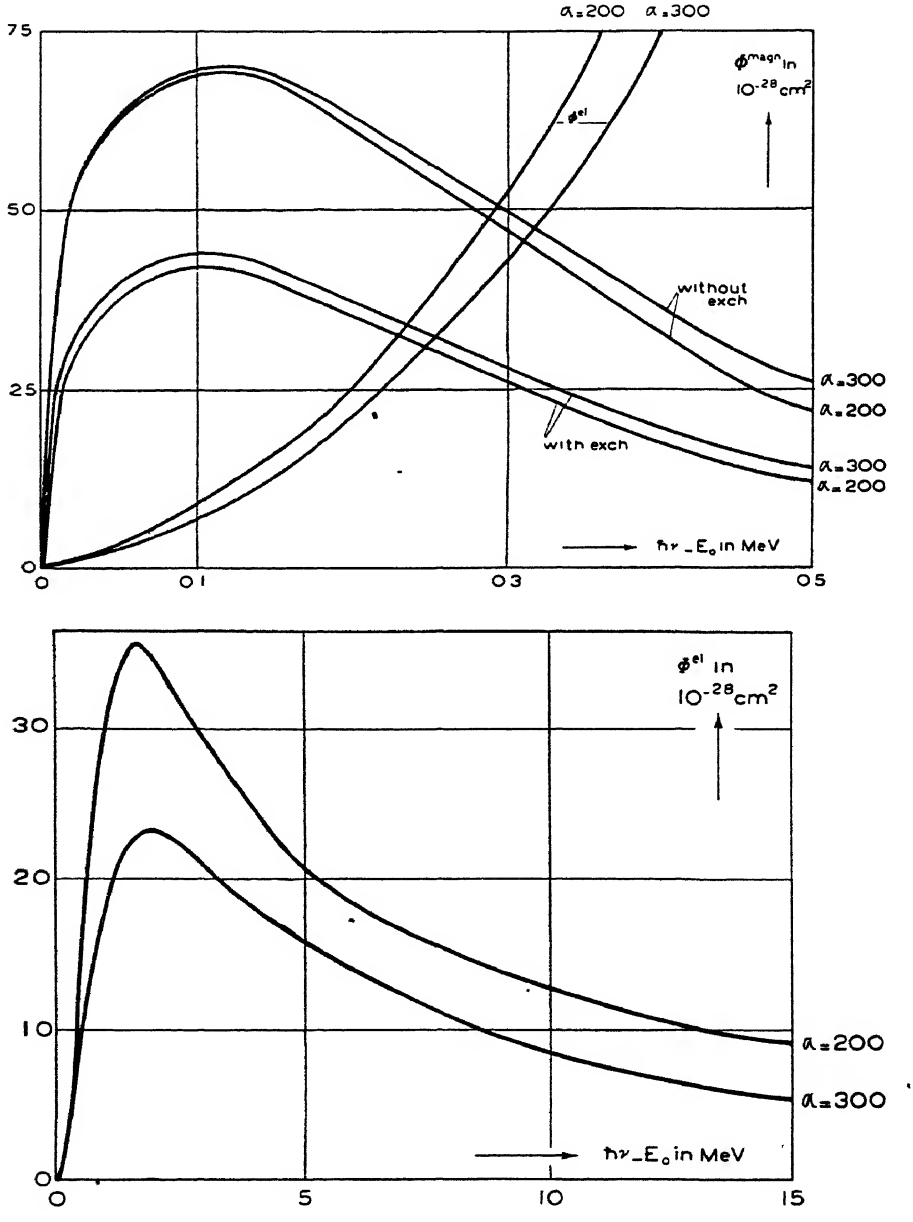

Fig. 8.34–1, 2. Photoelectric and photomagnetic disintegration cross-sections of the deuteron, according to PAIS [43]. (α means M_m/m.)

to the small number of tracks (about 60 in either case). The only indications about the parameter H which can be derived from this work are as follows:

$$\begin{aligned} \text{for} \quad & h\nu = 2{,}62 \text{ MeV}, \quad H < \tfrac{1}{4} \\ & h\nu = 3{,}0 \text{ MeV}, \quad H \lesssim 0{,}16. \end{aligned} \tag{26}$$

The direct measurement of the angular distribution of the photo-neutrons produced by 2,62 MeV (Th C″) γ-rays was first attempted by HALBAN [38] and repeated with an improved technique by GRAHAM and HALBAN [45]; the same method has also been applied by MYERS and VAN ATTA [42], who, however, used an ill-defined continuum of X-rays reaching up to 2,43 MeV. A glass sphere filled with heavy water is irradiated by a suitable source and the ejected neutrons are collected by a system of boron chamber detectors, the orientation of which relative to the axis source-sphere can be varied; rather large and not quite certain corrections must be applied to the measured intensities owing to the geometrical conditions and the scattering of the photo-neutrons within the sphere. The figure given for H by Myers and Van Atta, viz. $0{,}87 \pm 0{,}07$, has little significance; it may perhaps be compared with the theoretical values for the weighted average energy of 2,28 MeV. As to Halban's results for 2,62 MeV, they are unfortunately not concordant: his first experiments, with a sphere of 1 cm diameter, gave $H \approx 0{,}07 \pm 0{,}06$; the second, in collaboration with Graham, were performed with two spheres of diameters 1,6 and 0,77 cm, and the result, extrapolated to zero diameter was $H \approx 0{,}20 \pm 0{,}05$ (with some indications in favour of a low value). Further improvement of these difficult measurements is thus needed before any theoretical conclusion can safely be drawn.

At higher γ-ray energies, the angular distribution of the ejected nucleons is expected to follow very nearly the $\sin^2\vartheta$-law corresponding to the photoelectric process. For 6 MeV γ-rays, the distribution of the ejected protons has been investigated by the photographic plate method (*6.42*) (GIBSON, GREEN and LIVESEY [47]). The analysis of 60 tracks in the angular range $70° \ldots 180°$ gives a result compatible with the $\sin^2\vartheta$-law.

The total cross-section of the photodisintegration of the deuteron is hardly known at all with any precision. The only existing estimates are the following:

8.34-2. *Measurements of total photodisintegration cross-section*

Reference	$h\nu$ MeV	Total cross-section 10^{-28} cm²	Observations
HALBAN	2,62	13	Provisional value, not yet published
RICHARDSON and EMO [38]	3,0	(11)	May be wrong by a factor 2
ALLEN and SMITH [41]	6,2	11,6	

For the capture cross-section of neutrons of $2{,}2 \cdot 10^5$ cm/sec velocity, Hulthén finds the following results:

$$\text{for } M_m = 197m, \qquad G = 0{,}294 \cdot 10^{-24} \text{ cm}^2$$
$$\text{for } M_m = 295m, \qquad G = 0{,}303 \cdot 10^{-24} \text{ cm}^2;$$

the exchange effect is again entirely negligible. The comparison with the empirical value (*6.52*–18) would perhaps favour a high value of the meson mass. However, a slight modification of the ^1S wave-function, within the uncertainty of the empirical value (*6.22*–32) of the slow neutron-proton scattering cross-section (assuming 22 instead of 21 for S in units 10^{-24} cm^2), is sufficient to bring the value of G for $M_m = 197m$ to $0{,}308 \cdot 10^{-24}$ cm^2.

8.4. Central nuclear potential: summary of argument

At this stage, it will not be superfluous to review the line of argument followed in the preceding analysis of the properties of two-nucleon systems. This analysis starts from the assumption that we have to do with a *central nuclear potential*; it is further necessary to assume a definite form of distance dependence, which in the simplest cases involves two parameters, viz. the range constant and the strength. The available evidence then gives fairly complete information on the effective potentials in *triplet and singlet even configurations*; and in particular it supports the view that the analytical form of distance dependence and range value are the same in both cases, the strength alone being different (*8.2*).

The *proton-neutron potential* is markedly spin dependent. On the very natural assumption that the lowest stationary states to which this potential gives rise are S states (*6.11*), a large spin dependence is disclosed by the fact that the ground state of the deuteron belongs to the triplet system, as well as by the large cross-section of the proton for scattering or radiative capture of very slow neutrons. Indeed, the magnetic interaction between proton and neutron would depress the ^1S state below the ^3S state (*6.13*); whereas the large contribution of ^1S waves to the mentioned scattering and capture processes requires the existence of some (actual or virtual) ^1S level of very low energy (*6.23, 6.52*). The virtual character of this level, made probable by the study of the radiative capture process (*6.52*), is definitely established by the scattering properties of ortho- and para-hydrogen for slow neutrons (*6.33*). Thus, both the ^3S and ^1S effective potentials are attractive, but only the former leads to a stable stationary state of binding.

For a given law of distance dependence and a given value of the range of force, we can determine the strength of the ^3S potential from the observed binding energy of the corresponding stationary state; the strength of the ^1S potential then follows from the value of the scattering cross-section of protons for very slow neutrons (*6.431, 6.432*). In principle, the value of the range might also be derived from the scattering cross-sections for faster neutrons, especially the D(d, n)-neutrons of 2 ... 3 MeV energy;

but the data available at present still leave a large uncertainty about the quantity in question. They are at any rate compatible with the assumption that the range of the proton-neutron force is the same as that of the proton-proton force (*6.431*).

The *proton-proton potential* (exclusive of Coulomb force) can be determined with somewhat better accuracy, in range as well as in strength, from the experiments on proton-proton scattering; the meson type of distance dependence, however, constitutes a conspicuous exception: the theory of the scattering process proves to be very insensitive to large variations of the value of the meson mass (which fixes the potential parameters) (*7.13*). The ^1S proton-proton potential is found to be attractive, and in fact (on the *assumption* of equal ranges) very nearly the same as the ^1S proton-neutron potential (*8.1*): this property is well-established for the well and Gauss types of potentials, whereas the meson potential, owing to the above-mentioned uncertainty, hardly permits any significant conclusion for the time being.

Finally, there is good reason to believe that the *neutron-neutron potential* is identical with the proton-proton one (apart from the Coulomb energy). This conclusion is reached chiefly by a comparison of the masses of isobars with neutron excesses + 1 and — 1: it is verified that the mass difference of such isobaric pairs consists solely of the electrostatic energy of the excess proton in the latter nucleus (*3.3*). Altogether, from the evidence on even states one can surmise with great probability that in all (even or odd) configurations compatible with the exclusion principle the effective potential is (with a possible deviation of a few percent) *independent of the charges* of the interacting nucleons (*8.1*). The general form of the central potential operator is then given by (*8.2–1*); in this formula, two of the four parameters *a* remain as yet undetermined, due to insufficient evidence on the odd configurations.

In fact, although the study of the scattering of fast nucleons is in principle suited to give us direct information about the odd state potentials, it has not yet reached the extent and accuracy needed for this purpose (*7.131, 8.33*). In order to complete our knowledge of nuclear interactions, we must therefore turn to other phenomena, pertaining to more complex nuclear systems.